MATH 252: ABSTRACT ALGEBRA II
HOMEWORK #8

Problem 1 (DF 12.1.2). Let R be an integral domain and let M be an R-module. The *rank* of M is the maximal number of R-linearly independent elements of M.

(a) Suppose that M has rank n and that $x_1, \dots, x_n$ is any maximal set of R-linearly independent elements of M. Let $N = Rx_1 + \cdots + Rx_n$ be the R-submodule generated by $x_1, \dots, x_n$. Prove that N is isomorphic to R^n and that the quotient M/N is a torsion R-module. *[Hint: Show that the map $R^n \to N$ which sends the ith standard basis vector to x_i is an isomorphism of R-modules.]*

(b) Prove conversely that if M contains a submodule N that is free of rank n (i.e., $N \cong R^n$) such that the quotient M/N is a torsion R-module then M has rank n. *[Hint: Let $y_1, \dots, y_{n+1}$ be any $n+1$ elements of M. Use the fact that M/N is torsion to write $r_i y_i$ as a linear combination of a basis for N for some nonzero elements r_i of R. Use an argument like Proposition 12.1.3 to show that the $r_i y_i$, and hence also the y_i, are linearly dependent.]*

Problem 2 (DF 12.1.5). Let $R = \mathbb{Z}[x]$ and let $M = (2, x)$ be the ideal generated by 2 and x, considered as a submodule of R. Show that $\{2, x\}$ is not a basis of M. Show that the rank of M is 1 but that M is not free of rank 1.

Problem 3*. Let R be a PID and let M be a finitely generated torsion R-module. Show that there exists $y \in M$ such that $\operatorname{ann}(y) = \operatorname{ann}(M)$.

Problem 4. Let M be the $\mathbb{Z}$-module generated by x_1, x_2, x_3, x_4 subject to the relations

$$\begin{aligned} x_1 + 3x_2 - 9x_3 \phantom{{}+ 12x_4} &= 0 \\ x_1 + 3x_2 + 3x_3 + 12x_4 &= 0 \\ 2x_1 + 4x_2 + 2x_3 + 24x_4 &= 0 \end{aligned}$$

Give an explicit isomorphism of M to a direct sum of cyclic abelian groups. What are the invariant factors and elementary divisors of $\operatorname{Tor}(M)$?

Date: 23 March 2012; due Friday, 30 March 2012.