

A database of paramodular forms from quinary orthogonal modular forms

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ABSTRACT. We compute tables of paramodular forms of degree two and cohomological weight via a correspondence with orthogonal modular forms on quinary lattices.

1. Introduction

Number theorists have a longstanding tradition of making tables of modular forms (for a brief history, see [CMFs21, §2]), with myriad applications to arithmetic and geometry. Moving beyond GL_2 , there has been substantial interest in similar catalogues of Siegel modular forms, as computed first for $Sp_4(\mathbb{Z})$ by Kurokawa [Kur78] and then more systematically by Skoruppa [Sko92], Raum [Rau10], and many others. Moving beyond trivial level, we find several interesting families of congruence subgroups of symplectic groups. Among them, the paramodular groups have recently received considerable interest, owing in part to their agreeable theory of newforms [RS07] as well as applications in the Langlands program [BK19, Gro16]. Direct computations of paramodular forms have focused on the more troublesome case of (noncohomological) weight 2 [PY15, BPY16, PSY17] (analogous to weight 1 classical modular forms), working with Fourier expansions using clever and sophisticated techniques.

In this paper, we report on our computation of a moderately large database of paramodular forms for GSp_4 (i.e., degree 2), but via a complementary approach in weight ≥ 3 : we compute with algebraic modular forms [Gro99] on orthogonal groups of positive definite quadratic forms in five variables. Instead of working with Fourier expansions, we access only the underlying systems of Hecke eigenvalues (enough for Galois representations and L -function). An explicit correspondence between orthogonal and paramodular forms was first conjectured by Ibukiyama [IK17, Ibu19] and recently proven by van Hoften [vH21], Rösner–Weissauer [RW21], and Dummigan–Pacetti–Rama–Tornaría [DPRT21], with a certain pesky asterisk (see Conjecture 2.18). Our approach using quinary quadratic forms is analogous to the use of ternary quadratic forms to compute classical modular forms introduced by Birch [Bir91] (see also [Tor05, Ram14, Hei16, HTV]).

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Our algorithms involve lattice methods, as described by Greenberg–Voight [GV14] and further developed and investigated by Hein [Hei16], Ladd [Lad18], and Rama [Ram20, Ram20git].

The tables computed here supersede data computed by Rama–Tornaría [RT20] (squarefree level $N \leq 1000$): we compute in non-squarefull levels $N \leq 1000$, in higher weight, and with more Dirichlet coefficients (see Table 5 and the surrounding details). Our data is available online [ALRTV23], and it is being incorporated into the L -functions and Modular Forms Database (LMFDB) [LMFDB].

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2. Background

Siegel paramodular forms. We begin with a brief review of some background and setting up notation. Let $g \in \mathbb{Z}_{\geq 1}$, let $V := \mathbb{Q}^{2g}$, and equip V with the symplectic pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle: V \times V &\rightarrow \mathbb{Q} \\ \langle x, y \rangle &= x^\top \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} y \end{aligned} \quad (2.1)$$

where I is the $g \times g$ -identity matrix. Let $\mathrm{GSp}_{2g}(\mathbb{Q})$ be the group of symplectic similitudes of V , namely

$$\mathrm{GSp}_{2g}(\mathbb{Q}) := \{(\alpha, \mu) \in \mathrm{GL}_{2g}(\mathbb{Q}) \times \mathbb{Q}^\times : \langle \alpha x, \alpha y \rangle = \mu \langle x, y \rangle \text{ for all } x, y \in V\}. \quad (2.2)$$

Projection onto the second factor yields a character $\mu: \mathrm{GSp}_{2g}(\mathbb{Q}) \rightarrow \mathbb{Q}^\times$, whose kernel $\mathrm{Sp}_{2g}(\mathbb{Q}) := \ker \mu$ is the group of symplectic isometries of V . Let

$$\mathrm{GSp}_{2g}^+(\mathbb{Q}) := \{\alpha \in \mathrm{GSp}_{2g}(\mathbb{Q}) : \det(\alpha) > 0\} < \mathrm{GSp}_{2g}(\mathbb{Q}).$$

The Siegel upper half-space is

$$\mathcal{H}_g := \{z \in \mathrm{M}_g(\mathbb{C}) : z^\top = z \text{ and } \mathrm{Im}(z) > 0\}, \quad (2.3)$$

the space of $g \times g$ complex symmetric matrices whose imaginary part is positive definite. The group $\mathrm{GSp}_{2g}^+(\mathbb{Q})$ acts on \mathcal{H}_g by

$$\alpha z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = (az + b)(cz + d)^{-1}$$

for $\alpha \in \mathrm{GSp}_{2g}^+(\mathbb{Q})$ and $z \in \mathcal{H}_g$.

Let $\rho: \mathrm{GL}_g(\mathbb{C}) \rightarrow \mathrm{GL}(W)$ be a representation of $\mathrm{GL}_g(\mathbb{C})$ on a finite-dimensional \mathbb{C} -vector space W , and let $\Gamma \leq \mathrm{Sp}_{2g}(\mathbb{Q})$ be a discrete subgroup. The space $M_\rho(\Gamma)$ of Siegel modular forms of level Γ and weight ρ is the space of holomorphic functions $f: \mathcal{H}_g \rightarrow W$ such that

$$f(\gamma z) = \rho(cz + d)f(z) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \quad (2.4)$$

together with the requirement that f is holomorphic at the cusps of Γ if $g = 1$. When $g = 1$ and $\rho: \mathbb{C}^\times \rightarrow \mathbb{C}^\times$ is $x \mapsto x^k$, we write $M_k(\Gamma) = M_\rho(\Gamma)$; when $g = 2$ and $\rho = \rho_{k,j} := \det^k \otimes \mathrm{Sym}^j(\mathrm{std})$, we write $M_{k,j}(\Gamma) = M_\rho(\Gamma)$.

This paper is concerned with certain families of discrete subgroups for $g = 1$ and $g = 2$. Let N be a positive integer. We write $\Gamma_0(N) \leq \mathrm{Sp}_2(\mathbb{Q}) = \mathrm{SL}_2(\mathbb{Q})$ $K(N) \leq \mathrm{Sp}_4(\mathbb{Q})$ for the following groups.

$$\Gamma_0(N) := \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap \mathrm{SL}_2(\mathbb{Q}), \quad K(N) := \begin{pmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Q}).$$

The subgroup $K(N)$ is called the **paramodular subgroup** of level N .

In particular, we will be interested in the space $M_{k,j}(K(N))$ of **Siegel paramodular forms** of level N and weight $(k, j) \in \mathbb{Z}_{\geq 0}^2$, and we write $M_k(N) = M_k(\Gamma_0(N))$ for the space of classical modular forms of level N and weight k .

Hecke operators. For $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GSp}_4^+(\mathbb{Q})$ (with $a, b, c, d \in \mathbb{M}_2(\mathbb{Q})$), define a right action of $\mathrm{GSp}_4^+(\mathbb{Q})$ on functions $f : \mathcal{H}_2 \rightarrow \mathbb{C}$

$$(f|_{k,j}\alpha)(z) = \mu(\alpha)^{2k+j-3} \rho_{k,j}(cz+d)^{-1} f(\alpha z). \quad (2.5)$$

In particular, the condition (2.4) for $\rho_{k,j}$ is equivalent to $(f|_{k,j}\alpha)(z) = f(z)$ for all $\alpha \in K(N)$ (since $\mu(\alpha) = 1$).

The Siegel operator $\Phi : M_{k,j}(K(N)) \rightarrow M_{j+k}(N)$ is defined by

$$\Phi(f)(z) = \lim_{t \rightarrow \infty} f \begin{pmatrix} z & 0 \\ 0 & it \end{pmatrix},$$

and we say that $f \in M_{k,j}(K(N))$ is a **cusp form** if $\Phi(f|_{k,j}\gamma) = 0$ for all $\gamma \in \mathrm{Sp}_4(\mathbb{Q})$. We write $S_{k,j}(K(N))$ for the subspace of paramodular cusp forms.

For p prime, let $\lambda_{p,1} := \mathrm{diag}(1, 1, p, p) \in \mathrm{M}_4(\mathbb{Z})$ and $\lambda_{p,2} := \mathrm{diag}(1, p, p^2, p) \in \mathrm{M}_4(\mathbb{Z})$. For $i = 1, 2$, let $\{\alpha_{p,i,r}\}_r$ be representatives for $K(N) \backslash K(N)\lambda_{p,i}K(N)$, so that

$$K(N)\lambda_{p,i}K(N) = \bigsqcup_r K(N)\alpha_{p,i,r}; \quad (2.6)$$

we define the Hecke operator $T_{p,i} : M_{k,j}(K(N)) \rightarrow M_{k,j}(K(N))$ by

$$T_{p,i}(f) = \sum_r f|_{k,j}\alpha_{p,i,r}.$$

The operators $T_{p,i}$ do not depend on the choice of representatives $\{\alpha_{p,i,r}\}$, and they restrict to operators on cusp forms $S_{k,j}(K(N))$ [PSY18, Proposition 5.2], which we also denote $T_{p,i}$. The Hecke operators $T_{p,i}$ for $p \nmid N$ pairwise commute, and an eigenform $f \in S_{k,j}(K(N))$ is a common eigenvector.

Paramodular oldforms and newforms. The spaces $S_{k,j}(K(N))$ of paramodular cusp forms admit a theory of oldforms and newforms due to Roberts–Schmidt [RS06]. Let p be a prime number. Then there are three level-raising

operators, defined as follows:

$$\begin{aligned} \theta_{N,p}: S_{k,j}(K(N)) &\rightarrow S_{k,j}(K(Np)) \\ \theta_{N,p}(f) &= f|_{k,j} \begin{pmatrix} p & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \\ &+ \sum_{c=0}^{p-1} f|_{k,j} \begin{pmatrix} p & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & c & \\ & & & 1 \end{pmatrix} \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} \theta'_{N,p}: S_{k,j}(K(N)) &\rightarrow S_{k,j}(K(Np)) \\ \theta'_{N,p}(f) &= f|_{k,j} \begin{pmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & 1/p \end{pmatrix} + \sum_{c=0}^{p-1} f|_{k,j} \begin{pmatrix} 1 & & & c/(pN) \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \end{aligned} \quad (2.8)$$

and finally

$$\begin{aligned} \eta_{N,p}: S_{k,j}(K(N)) &\rightarrow S_{k,j}(K(Np^2)) \\ \eta_{N,p}(f) &= f|_{k,j} \begin{pmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & p^{-1} \end{pmatrix} \end{aligned} \quad (2.9)$$

These three operators $\theta_{N,p}$, $\theta'_{N,p}$, and $\eta_{N,p}$ commute with each other, with the level-raising operators for other primes, and with the operators $T_{\ell,i}$ for all $\ell \nmid Np$ [RS06, §4].

For $p \mid N$, the subspace of p -oldforms in $S_{k,j}(K(N))$ is defined as the sum

$$\begin{aligned} S_{k,j}^{p\text{-old}}(K(N)) &:= \theta_{N/p,p} S_{k,j}(K(N/p)) + \theta'_{N/p,p} S_{k,j}(K(N/p)) \\ &+ \eta_{N/p^2,p} S_{k,j}(K(N/p^2)) \end{aligned} \quad (2.10)$$

where the last term only occurs if $p^2 \mid N$; the space of oldforms is then

$$S_{k,j}^{\text{old}}(K(N)) := \sum_{p \mid N} S_{k,j}^{p\text{-old}}(K(N)). \quad (2.11)$$

The subspace of p -newforms $S_{k,j}^{p\text{-new}}(K(N)) \subseteq S_{k,j}(K(N))$ is the orthogonal complement of the subspace of p -oldforms under the Petersson inner product, and the subspace of newforms $S_{k,j}^{\text{new}}(K(N))$ is the orthogonal complement of the subspace of oldforms.

Definite orthogonal modular forms. As in the introduction, to compute paramodular forms in fact we compute certain orthogonal modular forms. These spaces of orthogonal modular forms are again defined by a level Λ and a weight W , as follows.

Let V be a finite-dimensional \mathbb{Q} -vector space equipped with a positive definite quadratic form $Q: V \rightarrow \mathbb{Q}$ and $T: V \times V \rightarrow \mathbb{Q}$ the associated bilinear form. The

orthogonal group of Q is the group of F -linear automorphisms of V which preserve Q , namely

$$O(Q) := \{g \in \mathrm{GL}(V) : Q(gx) = Q(x) \text{ for all } x \in V\};$$

elements of $O(Q)$ are called **isometries**. Let

$$\mathrm{SO}(Q) := O(Q) \cap \mathrm{SL}(V)$$

(the subgroup of isometries that preserve an orientation).

Let $\rho: \mathrm{SO}(Q) \rightarrow \mathrm{GL}(W)$ be a representation. We will sometimes refer to ρ by the underlying space W . If W is the irreducible representation of $\mathrm{SO}(Q)$ of highest weight λ , we write $W = W_\lambda$.

Let $\Lambda \subset V$ be a lattice, the \mathbb{Z} -span of a \mathbb{Q} -basis for V . Suppose that $Q(\Lambda) \subseteq \mathbb{Z}$, in which case we say Λ is **integral**. For two lattices $\Lambda, \Lambda' \subseteq V$, if there is an isometry $g \in O(V)$ such that $\Lambda' = g(\Lambda)$, we say that Λ and Λ' are **isometric**, and write $\Lambda \simeq \Lambda'$.

Making analogous definitions, we have $\Lambda_p := \Lambda \otimes \mathbb{Z}_p$ a \mathbb{Z}_p -lattice in $V \otimes \mathbb{Q}_p$. The **genus** of Λ is the set of lattices which are everywhere locally isometric to Λ , namely

$$\mathrm{Gen}(\Lambda) := \{\Lambda' \subset V : \Lambda_p \simeq \Lambda'_p \text{ for all primes } p\}. \quad (2.12)$$

There is a natural action of $O(V)$ on $\mathrm{Gen}(\Lambda)$ and the **class set** of Λ is the set of global isometry classes

$$\mathrm{Cls} \Lambda := O(V) \backslash \mathrm{Gen}(\Lambda).$$

By the geometry of numbers, the class set is finite; let $h = h(\Lambda) := \#\mathrm{Cls} \Lambda < \infty$. Let $\Lambda_1, \dots, \Lambda_h$ be a set of representatives for $\mathrm{Cls} \Lambda$, and let

$$\mathrm{SO}(\Lambda_i) := \{g \in \mathrm{SO}(V) : g(\Lambda_i) = \Lambda_i\} \quad (2.13)$$

The space of (special) orthogonal modular forms of level Λ and weight W is the space

$$M_W(\mathrm{SO}(\Lambda)) = \bigoplus_{i=1}^h W^{\mathrm{SO}(\Lambda_i)}. \quad (2.14)$$

If $W = W_{a,b}$, we also write $M_{a,b}(\mathrm{SO}(\Lambda)) := M_W(\mathrm{SO}(\Lambda))$. When W is trivial, $(a,b) = (0,0)$, and $M_W(\mathrm{SO}(\Lambda)) \simeq \mathbb{C}^{\#\mathrm{Cls} \Lambda} \simeq \mathbb{C}^h$.

For the correspondence with paramodular forms, we also need to define certain twists of these spaces by certain characters as follows. Let $p \mid N$ be prime, and let $R_p := \mathrm{rad}(\Lambda_p \otimes \mathbb{Z}/(2p)\mathbb{Z})$ be the **radical** of $\bar{\Lambda} := \Lambda_p \otimes \mathbb{Z}/(2p)\mathbb{Z}$, where

$$\mathrm{rad}(\bar{\Lambda}) := \{x \in \bar{\Lambda} : T(x, y) = 0 \text{ for all } y \in \bar{\Lambda}\}, \quad (2.15)$$

(extending T to $\bar{\Lambda}$). Since $g_p R_p = R_p$ for all $g_p \in \mathrm{Aut}(\Lambda_p)$, we may define a homomorphism $\vartheta_p: \mathrm{Aut}(\Lambda_p) \rightarrow \{\pm 1\}$ by $\vartheta_p(g_p) := \det(g_p|_{R_p})$. Given a unitary divisor $d \parallel N$ (so $d \mid N$ and $\gcd(d, N/d) = 1$), we define the associated **radical character** $\vartheta_d: \prod_{p \mid d} \mathrm{Aut}(\Lambda_p) \rightarrow \{\pm 1\}$ by $\vartheta_d = \prod_{p \mid d} \vartheta_p$ [DPRT21, §7]. Although ϑ_d is not defined on all of $\mathrm{SO}(V)$, as it is defined almost everywhere, using weak approximation we choose representatives Λ_i of the class set so that it is defined on $\mathrm{Aut}(\Lambda_i)$ for all i .

Finally, we write

$$M_{a,b}(\mathrm{SO}(\Lambda), \vartheta_d) := M_{W_{a,b} \otimes \vartheta_d}(\mathrm{SO}(\Lambda)). \quad (2.16)$$

Definite orthogonal methods for paramodular forms. The relation between modular forms on $\mathrm{SO}(5)$ and automorphic forms on $\mathrm{GSp}(4)$ with trivial central character is predicted by Langlands functoriality. An explicit correspondence was conjectured by Ibukiyama in [IK17, Ibu19] involving two steps: a correspondence between (para)modular forms of $\mathrm{GSp}(4)$ and its compact twist $\mathrm{GU}(2, B)$, where B is a definite quaternion algebra; and a correspondence between modular forms of $\mathrm{GU}(2, B)$ and those of $\mathrm{SO}(Q)$ for a suitable chosen quinary quadratic form Q . The first correspondence was proved by van Hoften [vH21] and Rösner–Weissauer [RW21], and extended by Dummigán–Pacetti–Rama–Tornaría [DPRT21], where the second correspondence was also proved.

More precisely, using [DPRT21, Theorem 9.9], we can compute the space $S_{k,j}^{\mathrm{new}}(K(N))$ of paramodular newforms of level N and weight (k, j) under the following assumptions:

- (1) There is a prime p_0 such that $p_0 \parallel N$ (i.e., p_0 exactly divides N); and
- (2) $k \geq 3$ and $j \in 2\mathbb{Z}_{\geq 0}$.

(When j is odd, the space is $\{0\}$.) Suppose that these conditions hold. Then [DPRT21, Theorem 5.14] there exists a unique genus of positive definite integral quinary quadratic forms of (half-)discriminant N with Eichler invariants -1 at p_0 and $+1$ at all other primes [DPRT21, §5]. We choose one quadratic form $Q = Q_{N,p_0}$ in the genus, corresponding to a lattice Λ , noting that the corresponding space of orthogonal modular forms is independent of this choice.

Following notation in [DPRT21, §8] and for notational convenience, write $a := k + j - 3$ and $b := k - 3$, noting $a \equiv b \pmod{2}$. Let $W_{a,b}$ be the corresponding weight representation of $\mathrm{SO}_5(\mathbb{C})$; and for each $d \parallel N$, let ϑ_d be the associated radical character. Let $M_{a,b}(\mathrm{SO}(\Lambda), \vartheta_d)$ be the space of orthogonal modular forms for Q_{N,p_0} with weight representation $W_{a,b}$ and character ϑ_d .

Now let $S_{k,j}^{p_0\text{-new}}(K(N))_{(\mathbf{G})} \subseteq S_{k,j}(K(N))$ be the space of p_0 -new paramodular cusp forms of general type (\mathbf{G}) . The condition (\mathbf{G}) of general type concerns the type of the automorphic representation, following Schmidt [Sch18, §1.1]: it excludes the forms of Saito–Kurokawa type (\mathbf{P}) on both sides, the forms of Yoshida type (\mathbf{Y}) appearing as orthogonal modular forms, and forms of Eisenstein type (those f whose automorphic representation π_f has local components $\pi_{f,p}$ isomorphic to a *noncuspidal* irreducible constituent of a representation parabolically induced from a proper Levi subgroup). See the section on detecting lifts below for further discussion. Similarly, let $M_{a,b}(\mathrm{SO}(\Lambda), \vartheta_d)_{(\mathbf{G})} \subseteq M_{a,b}(\mathrm{SO}(\Lambda), \vartheta_d)$ be the subspace of forms of general type (\mathbf{G}) .

Theorem 2.17 ([DPRT21, Theorem 9.6]). *There is an isomorphism of Hecke modules*

$$S_{k,j}^{p_0\text{-new}}(K(N))_{(\mathbf{G})} \simeq \bigoplus_{d \parallel N} M_{a,b}(\mathrm{SO}(\Lambda), \vartheta_d)_{(\mathbf{G})}.$$

In the isomorphism of Theorem 2.17, the paramodular forms corresponding to the summand $M_{a,b}(\mathrm{SO}(\Lambda), \vartheta_d)_{(\mathbf{G})}$ have their Atkin–Lehner signs ε_p determined by d : we have $\varepsilon_{p_0} = -1$ if and only if $p_0 \nmid d$, and for primes $p \neq p_0$ we have $\varepsilon_p = -1$ if and only if $p \mid d$ [DPRT21, Theorem 10.1].

In our analysis below, we can compute explicitly the forms of type (\mathbf{P}) and (\mathbf{Y}) using lifts. To obtain only forms of type (\mathbf{G}) , we need to identify those orthogonal

modular forms of Eisenstein type, which should be given as follows [DPRT21, Proposition 9.8 and the comment afterward].

Conjecture 2.18. *The subspace of $M_{a,b}(\mathrm{SO}(\Lambda), \vartheta_d)$ spanned by forms of Eisenstein type is the span of constant functions when $(a, b) = (0, 0)$ (i.e., $(k, j) = (3, 0)$) and is trivial otherwise.*

Proposition 2.19. *Conjecture 2.18 holds when N is squarefree.*

PROOF. The same argument (comparing dimensions) in [DPRT21, Proposition 9.8] applies. \square

Thus, the correctness of our computations in nonsquarefree level are conditional on the truth of Conjecture 2.18: if this conjecture is false, it would mean that we have produced a counterfeit form which appears to be of type **(G)**. We expect this never to happen, and we hope soon that this will be decisively resolved.

3. Algorithmic comments

Algorithm overview. Algorithms to compute with orthogonal modular forms using lattice methods were exhibited by Greenberg–Voight [GV14]; a recent overview with concrete examples is given by Assaf–Fretwell–Ingalls–Logan–Secord–Voight [AFILSV22, §3]. These algorithms take as input an integral, positive definite quadratic form on a lattice Λ and compute the action of Hecke operators on spaces of functions on the class set of Λ , with values in a weight representation. The Hecke operators are computed as p -neighbors, after Kneser, using an algorithm to test lattice isomorphism due to Plesken–Souvignier. For evaluating the Hecke operator $T_{p,1}$, the running time complexity is dominated by $O(hp^3)$ isometry tests, where h is the class number of the lattice; for $T_{p,2}$, it is dominated by $O(hp^4)$ isometry tests.

In this way, we can compute the Hecke operators for all good primes $p \nmid N$ and bad primes $p \parallel N$ [RT20], giving the Euler factors at these primes. See the section below on Euler factors for bad primes $p^2 \mid N$.

In order to compute all Dirichlet coefficients a_n for $n \leq M$ for the L -series attached to f , we need to compute the Hecke operators $T_{p,1}$ for $p \leq M$ and the Hecke operators $T_{p,2}$ for $p \leq \sqrt{M}$. Hence the running time complexity for computing all of the first M coefficients of the L -series is dominated by a total of $O\left(\frac{hM^4}{\log M}\right)$ isometry tests.

Implementation. For reliability, we carried out and compared two separate implementations to compute the data, one in C and one in PARI/GP [PARI/GP]. Eventually, these gave the same output. The code can be found, respectively, at <https://github.com/assaferan/omf5> and at <https://gitlab.fing.edu.uy/grama/quinary>. We have also used SageMath [S⁺09] for auxiliary operations on the data we produced.

Remark 3.1. In [AFILSV22], the authors present a Magma [Magma] implementation, to be found at <https://github.com/assaferan/ModFrmAlg>, which agrees with the other two. It is more general (working in arbitrary dimension and over a totally real base field); our implementation is specialized, and more efficient for this case.

Algorithmic improvements. In order to make the calculations of Hecke operators mentioned in the previous section more efficient, it is possible to take advantage of the action of the isometry group of the lattice. Indeed, two p -isotropic vectors in the same orbit of the isometry group $\text{Aut}(\Lambda)$ will produce the same target lattice when applying the p -neighbor relation. It is also possible to save on isometry testing via taking the first few entries of the theta series for the lattice.

To carry out this idea (which has been observed before), we present a few algorithmic improvements to further speed up the computation by a constant factor.

- (1) Instead of precomputing all the orbits of isotropic vectors under $\text{Aut}(\Lambda)$, we order the isotropic vectors in a lexicographical order. Given a \mathbb{Z}_p -isotropic vector v , we compute its orbit under $\text{Aut}(\Lambda)$, and proceed with the computation only when it is the minimal vector in its orbit.

Using the automorphism group speeds up the running time by the average size of an orbit, hence by a factor of $\#\text{Aut}(\Lambda)$. Note that even over all rank 5 lattices Λ , the sizes $\#\text{Aut}(\Lambda)$ remain bounded, hence this is only a constant factor improvement. Precomputation of the orbits [AFILSV22, §3] also yields this improvement, but at the cost of using $O(p^3)$ memory for $T_{p,1}$ and $O(p^4)$ memory for $T_{p,2}$; here we reduce these memory costs without performing additional isometry tests or short vector computations.

- (2) We precompute the automorphism group of all lattices in the genus, and their conjugations into a single quadratic space, saving the cost of conjugation when computing the spinor norm. The factor that this improvement yields depends on the ratio T_{conj}/T_θ between the time spent on conjugation of matrices and the time spent on computing short vectors. In practice, in our C implementation, this improves the running time by an average factor of 1.3.
- (3) If we have a list of the lengths of short vectors of the lattice we need only check for isometry with those members of the genus with the same list. For a given genus we measure the cost T_{isom} of isometry testing and the cost $T_\theta(B)$ of computing the short vectors of length up to B , and we choose B that optimizes the total cost $\alpha(B)T_{\text{isom}} + T_\theta(B)$, where $\alpha(B)$ is the average number of collisions, with the lattices in the genus averaged by the size of their automorphism groups. This follows from the fact that the frequency of appearance of a lattice Λ as a p -neighbor is inversely proportional to $\#\text{Aut}(\Lambda)$, as proven by Chenevier [Che22]. In practice, in our C implementation, this yields an average speed up factor of 6.84.

Example 3.2. For the case $N = 61$ with trivial weight, in the C implementation, it takes 113 s to compute all eigenvalues for $p < 100$ without these improvements. With all of them put together it takes 10.28 s, giving a speedup factor of 11.

Example 3.3. For the case $N = 61$, in the Pari/GP implementation, it takes 3881 s to compute a row for the space of quinary modular forms for the representation $(a, b) = (1, 0)$ (corresponding to $(k, j) = (4, 0)$) and the character θ_{61} without the isometry improvement. With the isometry improvement it took 451 s for a row that corresponds to a quadratic form with 96 automorphisms. And 859 s for a row that corresponds to a quadratic form with 12 automorphisms. This gives us a speedup factor of 8.6 and a factor of 4.5 respectively.

Example 3.4. Again for the case $N = 61$, in the Pari/GP implementation, it takes 4440 s to compute all a row for the space of quinary modular forms for weight $(a, b) = (1, 1)$ and the character θ_{61} , without the isometry improvement. With the isometry improvement it took 779 s for a row that corresponds to a quadratic form with 96 automorphisms. And 1188 s for a row that corresponds to a quadratic form with 12 automorphisms. This gives us a factor of 5.7 and a factor of 3.7, respectively.

Detecting lifts. Among the orthogonal forms we compute we have forms that arise from lifts (endoscopy), just like Eisenstein series in the classical case. Since those lifts are classified and can be computed in other ways, we focus on what remains on computing newforms of type **(G)**.

To discard forms of type **(P)** corresponding to Saito–Kurokawa lifts, a single good Euler factor is enough: a form of type **(P)** will have a Satake parameter $p^{1/2}$ that cannot otherwise appear by the Ramanujan conjecture for non-CAP forms [Wei09]. The multiplicity for forms of type **(P)** is given by [RS07, Theorem 5.5.9]; we do not need this as the stated criterion will filter out old and new Saito–Kurokawa lifts alike.

To discard forms of type **(Y)** corresponding to Yoshida lifts, it suffices to find that a single good Euler factor is irreducible. If all the computed degree 4 Euler factors are product of two degree 2 factors, we look in tables to conjecturally identify the form as a Yoshida lift, as we know exactly which Yoshida lifts should appear [DPRT21, Proposition 9.1]; the radical character is determined by the Atkin-Lehner signs which are given by [DPRT21, Theorem 10.1 (3)]. The multiplicity of each Yoshida lift is given by [RS07, Theorem 7.5.6] with Atkin-Lehner signs as in [RS07, Table (5.49)]; those are local results so we have to take the product over primes with Eichler invariant $+1$ (for primes with Eichler invariant -1 —i.e., p_0 —Yoshida lifts are always new and have local multiplicity one). To identify the lifts, a simple inspection and comparison of the traces often suffices, otherwise we use the complete Euler factors.

Newforms and oldforms. In all cases, we have a good guess as to what is new and old; computing inductively we can verify that a form is not an oldform. However, oldforms may appear in the space of orthogonal modular forms with multiplicity.

Using [RS07, Theorem 7.5.6] we can compute the multiplicity for each oldform with Atkin-Lehner signs given by [RS07, Table (5.49)]. Again we must take the product over primes with Eichler invariant $+1$; for primes with Eichler invariant -1 the forms are always new with multiplicity one and the Atkin-Lehner sign is changed (see [DPRT21, Theorem 10.1 (1)]).

With these multiplicities, we can certify the oldform and newform decomposition by computing more Hecke eigenvalues until the dimension of a candidate oldspace with given eigenvalues matches the multiplicity.

4. Running the calculation

Example. We begin with an example to illustrate the calculations we performed.

Consider the space $S_{3,0}(K(312))$. Since $312 = 2^3 \cdot 3 \cdot 13$, we choose $p_0 = 13$, and produce a lattice Λ with (half-)discriminant 312 and Eichler invariant -1 at

13 and +1 at 2 and 3. Explicitly we take $\Lambda = \mathbb{Z}^5$ equipped with the quadratic form having the following Gram matrix:

$$\begin{pmatrix} 2 & 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & -1 & 1 \\ 1 & -1 & 4 & -1 & 0 \\ 0 & -1 & -1 & 6 & 0 \\ 1 & 1 & 0 & 0 & 12 \end{pmatrix}$$

A quick computation shows that the class set of Λ has cardinality 15. Computing the spaces $M_{0,0}(\mathrm{SO}(\Lambda), \theta_d)$ for squarefree $d \mid 312$ we find that their dimensions are given as in Table 1.

d	1	2	3	6	13	26	39	78	Total
(G)-new	1	0	0	5 = 3 + 1 + 1	0	1	4 = 3 + 1	0	11
(G)-old	1	1	1	1	1	2	0	1	8
(P)-new	2	0	0	3	0	2	2	0	9
(P)-old	8	0	0	2	0	7	2	0	19
(Y)-new	1	0	0	1	0	1	3	0	6
(Y)-old	1	1	0	0	1	1	0	0	4
Total	14	2	1	12	2	14	11	1	57

TABLE 1. Dimensions of subspaces of $M_{0,0}(\mathrm{SO}(\Lambda), \theta_d)_{(\Lambda)}$ for $d \mid 312$

As explained in the previous section, we calculate the space of Yoshida lifts:

$$\begin{aligned} M_{0,0}^{\mathrm{new}}(\mathrm{SO}(\Lambda))_{(\mathbf{Y})} &\simeq (S_2^{\mathrm{new}}(\Gamma_0(26)) \otimes S_4^{\mathrm{new}}(\Gamma_0(12))) \\ &\quad \oplus (S_2^{\mathrm{new}}(\Gamma_0(39)) \otimes S_4^{\mathrm{new}}(\Gamma_0(8))) \\ &\quad \oplus (S_2^{\mathrm{new}}(\Gamma_0(52)) \otimes S_4^{\mathrm{new}}(\Gamma_0(6))), \end{aligned} \quad (4.1)$$

leading to the corresponding dimension counts in Table 1. Furthermore, the only Yoshida lifts that occur as oldforms are the images of the forms in

$$M_{0,0}^{\mathrm{new}}(\mathrm{SO}(\Lambda_{156}))_{(\mathbf{Y})} \simeq S_2^{\mathrm{new}}(\Gamma_0(26)) \otimes S_4^{\mathrm{new}}(\Gamma_0(6))$$

under the level-raising operators $\theta_2, \theta'_2: S_{3,0}(K(156)) \rightarrow S_{3,0}(K(312))$.

Similarly, we find that

$$M_{0,0}^{\mathrm{new}}(\mathrm{SO}(\Lambda))_{(\mathbf{P})} \simeq S_4^{\mathrm{new},-}(\Gamma_0(312)) \oplus S_4^{\mathrm{new},+}(\Gamma_0(24)),$$

(plus and minus signs for the Atkin–Lehner involution), and

$$\begin{aligned} M_{0,0}^{\mathrm{old}}(\mathrm{SO}(\Lambda))_{(\mathbf{P})} &\simeq \bigoplus_{d \mid 24, d \neq 24} S_4^{\mathrm{new},-}(\Gamma_0(13d)) \oplus S_4^{\mathrm{new},+}(\Gamma_0(d)) \\ &\quad \oplus \bigoplus_{d \mid 6} S_4^{\mathrm{new},-}(\Gamma_0(13d)) \oplus S_4^{\mathrm{new},+}(\Gamma_0(d)); \end{aligned} \quad (4.2)$$

this yields further dimension counts in Table 1.

Finally, using our precomputed data from lower levels, we find that

$$\dim S_{3,0}(K(156))_{(\mathbf{G})} = 3, \quad \dim S_{3,0}(K(104))_{(\mathbf{G})} = 1.$$

Using the more precise data of their Atkin–Lehner eigenvalues and applying the corresponding level-raising operators, we find that they contribute to an 8-dimensional

subspace, distributed between the different subspaces as described in the second row of Table 1.

The subspace orthogonal to all of the above is the space of Siegel paramodular newforms of type (\mathbf{G}) [DPRT21, Theorem 9.9], given in the top row. For these, we have also included in the table the decomposition of the subspaces into Hecke irreducible submodules, i.e., Galois orbits of paramodular newforms.

It then follows that $\dim S_{3,0}^{\text{new}}(K(312))_{(\mathbf{G})} = 11$, and the space is composed of 7 Galois orbits of newforms, out of which 5 newforms have rational eigenvalues and the 2 remaining orbits have dimension 3. Table 2 gives the dimensions of various subspaces of $S_{3,0}(K(312))$.

type	new	old	total
(\mathbf{G})	11	8	19
(\mathbf{P})	9	19	28
Total	20	27	47

TABLE 2. Dimensions of subspaces of $S_{3,0}(K(312))$

Table 3 gives the dimensions of the cuspidal new subspaces of automorphic type (\mathbf{G}) with specified eigenvalues for the Atkin-Lehner operators and the Fricke involution.

2	3	13	Fricke	dimension
+	+	-	-	1
+	-	+	-	4
-	+	+	-	1
-	-	-	-	5
Plus space			+	0
Minus space			-	11

TABLE 3. Dimensions of Atkin-Lehner subspaces of $S_{3,0}^{\text{new}}(K(312))_{(\mathbf{G})}$

Finally, since we are able to compute the Hecke eigenvalues at good primes for each of the newforms, we can decompose $S_{3,0}^{\text{new}}(K(312))_{(\mathbf{G})}$ to newform subspaces. Table 4 lists the Galois orbits of the Hecke eigenforms in this space, giving rise to such a decomposition.

Here, the field 3.3.961.1 is given by its LMFDB label: it is defined by a root of the polynomial $x^3 - x^2 - 10x + 8$.

Data and running time. The data is available online [ALRTV23]. We computed the spaces of paramodular forms of level N and weight (k, j) , the Hecke eigenforms and the eigenvalues of the Hecke operators in the following ranges:

- $(k, j) = (3, 0)$, $D = N \leq 1000$, good $T_{p,i}$ with $p^i < 200$
- $(k, j) = (4, 0)$, $D = N \leq 1000$, good $T_{p,1}$ with $p < 100$, good $T_{p,2}$ with $p < 30$
- $(k, j) = (3, 2)$, $D = N \leq 500$, good $T_{p,1}$ with $p < 100$, good $T_{p,2}$ with $p < 30$

dimension	field	Traces				A-L signs		
		a_5	a_7	a_{11}	a_{17}	2	3	13
1	\mathbb{Q}	-1	-13	-6	63	+	+	-
1	\mathbb{Q}	-11	3	-16	3	+	-	+
1	\mathbb{Q}	1	-15	14	135	-	+	+
1	\mathbb{Q}	2	-6	-52	44	-	-	-
1	\mathbb{Q}	-13	-3	-4	-37	-	-	-
3	3.3.961.1	-1	-25	-24	-71	+	-	+
3	3.3.961.1	-12	28	2	-90	-	-	-

TABLE 4. Decomposition of $S_{3,0}^{\text{new}}(K(312))_{((\mathbf{G}))}$ into newform subspaces

(k, j)	Newspaces			Newforms		
	sqfree N	nonsqfree N	total	sqfree N	nonsqfree N	total
(3, 0)	2 764	4 820	7 584	52 181	23 853	76 034
(3, 2)	1 363	3 072	4 435	72 551	29 226	101 777
(4, 0)	2 856	7 783	10 639	287 974	132 380	420 354

TABLE 5. Newspace and newform data computed (abbreviating squarefree to “sqfree”)

The total counts are summarized in Table 5.

For squarefree levels, we use formulas from Ibukiyama–Kitayama [IK17] and Ibukiyama [Ibu19] to double-check our results.

We compare the performance of the two implementations (Pari and C) by comparing the computation time of a single row of the matrix representing the Hecke operator $T_{p,1}$ on the spaces $S_{3,0}(K(N))$ for various values of N and p . We note that this step is the bottleneck of the computation, hence this comparison provides ample evidence for the differences in performance as the parameters scale.

$p \backslash N$	61		167		262		334	
	C	Pari	C	Pari	C	Pari	C	Pari
31	0.05 s	1.27 s	0.45 s	1.31 s	0.37 s	1.58 s	0.49 s	2.46 s
41	0.10 s	2.86 s	0.80 s	2.84 s	0.94 s	3.56 s	1.02 s	5.53 s
101	1.09 s	41.3 s	6.71 s	41.1 s	7.86 s	50.8 s	12.6 s	79.8 s
127	2.00 s	81.8 s	10.1 s	81.4 s	14.0 s	100 s	20.1 s	158 s
199	6.49 s	312 s	37.16 s	312 s	42.16 s	384 s	75.06 s	607 s

TABLE 6. Timings for computing $T_{p,1}$

For Hecke irreducible spaces of dimension ≥ 20 , we only store the traces; to avoid painful calculations in an extension field, we compute modulo a prime \mathfrak{p} of degree 1 which is large enough, and reconstruct.

The total data takes approximately 200 MB of disk space and took a total of 4575 hours of CPU time on a standard processor (Intel Xeon Gold 6240 CPU, 2.60GHz). Computing the eigenvalues for $p < 200$ took a total of 13869 hours of CPU time on a standard processor (AMD Ryzen Threadripper 2970WX CPU, 2.20

GHz) equipped with Ubuntu 22.04.2 operating system, achieving maximal memory usage of 303276 KB.

Bad Euler factors. When $p \parallel N$, the local Euler factor can be computed via neighbors (see section 3). When $p^2 \mid N$, the local Euler factor has degree at most 2 and is given explicitly by Roberts–Schmidt [RS07, Theorem 7.5.3] in terms of eigenvalues for a pair of Hecke operators which correspond to p and p^2 -neighbors [DPRT21, Proposition 8.5] (extended in the same way for $p \neq p_0$).

The first such bad factor occurs at level $76 = 2^2 \cdot 19$: the bad Euler factors are

$$\begin{aligned} L_2(f_{76}, X) &= 1 + 5X + 2^3 X^2 \\ L_{19}(f_{76}, X) &= (1 + 19X)(1 - 50X + 19^3 X^2). \end{aligned}$$

(Since $19 \parallel 76$, we compute the latter without any guesswork.) The next one occurs at level $96 = 2^5 \cdot 3$, with

$$\begin{aligned} L_2(f_{96}, X) &= 1 + 4X + 2^3 X^2 \\ L_3(f_{96}, X) &= (1 - 3X)(1 + 8X + 3^3 X^2). \end{aligned}$$

We used our truncated Dirichlet series to check that the functional equation is satisfied up to 22 decimal digits of precision in each case, using both Magma and Pari/GP. (In practice, for $p^2 \mid N$ we found easier to guess the appropriate Hecke operator via reconstruction by checking that the functional equation for the L -function is satisfied—ruling out all possibilities but one.)

In order to rigorously verify that the functional equation is satisfied to a given precision, one would need the first $C\sqrt{N}$ embedded Dirichlet coefficients (with sufficient precision) for a constant C (see below), carrying out all floating-point calculations using rigorous error bounds and interval arithmetic. A generic library to carry out such computations, due originally to David Platt [CP19], has been developed; it includes functionality to estimate the constant C as a function of the motivic data. However, for L -functions of Siegel paramodular forms, it yields an estimated constant of $C \approx 138.84$, meaning one needs the Dirichlet coefficients for $p^n < 1211$ for f_{76} and for $p^n < 1361$ for f_{96} . This is in stark contrast to the small constant $C = 5$ which suffices for genus 2 curves [BSSVY16] or $C = 0.08k \log k + 24$ sufficient for classical modular forms of weight k [CMFs21]. Therefore, at the moment we satisfy ourselves with the heuristic of checking the functional equation for the truncated Dirichlet series that we have computed.

Statistics. Having compiled such a database with substantial amount of information, using the LMFDB [LMFDB], we are able to provide some arithmetic statistics of interest, examples of which we present in Table 7 and Table 8, illustrating the distribution of the dimension of the Galois orbit of a newform of type (\mathbf{G}) , and the distribution of the automorphic type, respectively, as the level increases.

Interesting examples. When compiling a large amount of modular forms in a database, it is easy to view them as a whole. However, each form has its own unique features and set of characteristics that make them different from any other form. We seize the opportunity to point out several specific forms of interest, and refer to their rich history in the literature and in previous works. We focus on weight $(k, j) = (3, 0)$ in this section.

level	dimension							Total
	1	2	3	4	5-10	11-20	21-266	
1-100	14 0.18%	1 0.01%						15 0.20%
101-300	258 3.40%	132 1.74%	32 0.42%	42 0.55%	59 0.78%	11 0.15%	1 0.01%	535 7.05%
301-600	864 11.40%	384 5.06%	183 2.41%	156 2.06%	358 4.72%	164 2.16%	155 2.04%	2264 29.85%
601-999	1436 18.93%	693 9.14%	399 5.26%	275 3.63%	684 9.02%	474 6.25%	809 10.67%	4770 62.90%
Total	2572 33.91%	1210 15.95%	614 8.10%	473 6.24%	1101 14.52%	649 8.56%	965 12.72%	7584

TABLE 7. Distribution of levels and dimensions in weight $(3, 0)$

level	automorphic type			Total
	(G)	(P)	(Y)	
1-100	15 0.10 %	169 1.15 %	11 0.07 %	195 1.29 %
101-300	535 3.53 %	787 5.19 %	166 1.09 %	1488 9.81 %
301-600	2264 14.93 %	1613 10.64 %	709 4.68 %	4586 30.25 %
601-999	4770 31.46 %	2476 16.33 %	1647 10.86 %	8893 58.65 %
Total	7584 50.02 %	5045 33.27 %	2533 16.71 %	15162

TABLE 8. Distribution of automorphic type in weight $(3, 0)$

- The smallest level in which a paramodular form of weight $(3, 0)$ and automorphic type (**G**) occurs is 61. This form has Hecke eigenvalues $a_2 = -7, a_3 = -3, a_5 = 3, a_7 = -9, \dots$, and was first described in [PY15].
- The smallest level in which there is a paramodular newform of weight $(3, 0)$ with negative sign in the functional equation is 167. This form has Hecke eigenvalues $a_2 = -8, a_3 = -10, a_5 = -4, a_7 = -14, \dots$. It was found in [RT20], paving the way for the correspondence presented in [DPRT21].
- The first level to exhibit a non-rational paramodular newform of type (**G**) is 97. The Hecke field is $\mathbb{Q}(\sqrt{5})$, and the first Hecke eigenvalues are $a_2 = -5 + \varphi, a_3 = -4\varphi, a_5 = -8 + 12\varphi, a_7 = 9 - 2\varphi, \dots$, where $\varphi = \frac{1+\sqrt{5}}{2}$.
- The paramodular newform of type (**G**) with the largest dimension of its Galois orbit in our database occurs in level 997, and its dimension is 266.
- There are three paramodular newforms of type (**G**) attaining the largest index of the Hecke ring inside the maximal order of the Hecke field among all paramodular forms computed. These forms appear in levels 384, 448 and 483, with the two first ones having Hecke field $\mathbb{Q}(\zeta_7)^+$, and the third

$\mathbb{Q}(\sqrt{5})$. In all three cases, the index of the Hecke ring inside the maximal order is 32.

Future directions. We close with some possible directions for future work.

- An immediate goal is to gather data for higher weights (k, j) in a range where computation is feasible and find the Euler factors at the bad primes to have the complete eigensystem for every form.
- To understand oldforms and newforms, another direction we may proceed is to implement the level-raising operators between the spaces of orthogonal modular forms and relating them to the corresponding level-raising map for paramodular forms given in (2.7)–(2.9). This could be used to construct directly the spaces of newforms, without computing the exact multiplicity in which each oldform appears (or giving an independent check).
- Finally, we also hope to complete the certification of the L -series for the newforms we have found, perhaps by further speeding up the computation, in order to allow us to compute a sufficient amount of Hecke eigenvalues.

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