MONODROMY GROUPS OF JACOBIANS WITH DEFINITE QUATERNIONIC MULTIPLICATION

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ABSTRACT. Let A be an abelian variety over a number field. The connected monodromy field of A is the minimal field over which the images of all the ℓ-adic torsion representations have connected Zariski closure. We show that for all even $g \geq 4$, there exist infinitely many geometrically nonisogenous abelian varieties A over $\mathbb{Q}$ of dimension g where the connected monodromy field is strictly larger than the field of definition of the endomorphisms of A. Our construction arises from explicit families of hyperelliptic Jacobians with definite quaternionic multiplication.

CONTENTS

1. INTRODUCTION

1.1. **Motivation.** Let K be a number field with algebraic closure K^{al} and absolute Galois group $\mathrm{Gal}_K := \mathrm{Gal}(K^{\mathrm{al}} \,|\, K)$. Let A be an abelian variety defined over K with $g := \dim A$, and write $A^{\mathrm{al}} := A \times_K K^{\mathrm{al}}$. Let $K(\mathrm{End}\, A)$ be the minimal extension of K over which the geometric endomorphism ring $\mathrm{End}\, A^{\mathrm{al}}$ is defined, called the endomorphism field of A. By a result of Silverberg [Sil92, Proposition 2.2], for all $m \geq 3$ we have

$$(1.1.1) \qquad\qquad K(\mathrm{End}\, A) \subseteq K(A[m]),$$

where $K(A[m])$ is the field generated by the m-torsion of A.

Let ℓ be prime. The rational ℓ-adic Tate module $V_\ell(A) := T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ of A has a natural Gal_K action, furnishing an ℓ-adic representation $\rho_{A,\ell} \colon \mathrm{Gal}_K \to \mathrm{GL}(V_\ell(A))$. Let $G_{A,\ell}$ be the Zariski closure of the image of $\rho_{A,\ell}$, a linear algebraic group over $\mathbb{Q}_\ell$ called the ℓ-adic monodromy group of A. Let $G^0_{A,\ell} \leq G_{A,\ell}$ be the connected component of the identity.

Serre [Ser13, no. 133] (see also Larsen–Pink [LP12, Propositions (6.12) and (6.14)]) proved that there exists a finite group Φ_A and a surjective group homomorphism

$$(1.1.2) \qquad\qquad \varepsilon_A \colon \operatorname{Gal}_K \to \Phi_A,$$

such that the induced map to the component group of $G_{A,\ell}$

$$(1.1.3) \qquad\qquad \varepsilon_{A,\ell} \colon \operatorname{Gal}_K \to \pi_0(G_{A,\ell}) = \frac{G_{A,\ell}(\mathbb{Q}_\ell)}{G_{A,\ell}^0(\mathbb{Q}_\ell)}$$

factors through ε_A via a canonical group isomorphism $\Phi_A \simeq \pi_0(G_{A,\ell})$. The finite extension of K cut out by $\ker \varepsilon_A$ is accordingly denoted $K(\varepsilon_A)$ (also denoted K_A^{conn}) and called the connected monodromy field of A. In other words, $K(\varepsilon_A)$ is the minimal extension over which the ℓ-adic monodromy groups of the base change of A become connected for all primes ℓ. By a theorem of Larsen–Pink [LP97, Theorem 0.1], the connected monodromy field is also described as the intersection over all primes ℓ of the fields generated by all ℓ-power torsion points of A:

$$(1.1.4) \qquad\qquad K(\varepsilon_A) = \bigcap_\ell K(A[\ell^\infty]).$$

Combining (1.1.1) and (1.1.4), we see that

$$(1.1.5) \qquad\qquad K(\operatorname{End} A) \subseteq K(\varepsilon_A).$$

We are therefore led to the following (folklore) question.

Question 1.1.6. *Let A be an abelian variety over a number field K. Under what conditions on A does the equality $K(\operatorname{End} A) = K(\varepsilon_A)$ hold?*

In other words, when is it the case that the ℓ-adic monodromy groups are connected whenever all endomorphisms of A are defined? Banaszak–Kedlaya [BK15, Theorem 6.10] proved that equality holds when $\dim A \leq 3$. Of course, equality holds for any principally polarized A with maximal ℓ-adic image $\operatorname{GSp}_{2g}(\mathbb{Z}_\ell)$ for some prime ℓ, since then the ℓ-adic monodromy group $G_{A,\ell} = \operatorname{GSp}_{2g,\mathbb{Q}_\ell}$ is already connected; hence 'most' abelian varieties A over K have $K(\operatorname{End} A) = K(\varepsilon_A) = K$.

1.2. **Results.** In this paper, we construct (to our knowledge, the first) examples where the connected monodromy field is strictly larger than the endomorphism field.

Theorem 1.2.1. *For all even $g \geq 4$, there exist infinitely many geometrically nonisogenous abelian varieties A over $\mathbb{Q}$ with $\dim A = g$ and $\mathbb{Q}(\operatorname{End} A) \subsetneq \mathbb{Q}(\varepsilon_A)$.*

Recently, Banaszak–Cantoral-Farfán [BCF22, Corollary 3.7] showed that, for a class of abelian varieties that includes ours, the relative degree $[K(\varepsilon_A) : K(\operatorname{End} A)]$ is at most 2. (For a precise statement, see Theorem 2.3.4 below.) Theorem 1.2.1 implies that this bound is in general sharp within this class.

We now briefly indicate our construction. We find our examples among those with geometric endomorphism algebra of Type III (definite quaternionic multiplication) in the

Albert classification. More precisely, we work with Jacobians of curves whose automorphism group contains the quaternion group Q_8 of order 8.

For $g \geq 4$ even, let $d := g/2 - 1$, and define the family of nice curves with affine model

$$(1.2.2) \qquad \mathsf{C}^{(g)}: \; y^2 = x(x^4 - 1)\left(x^{2g-4} + 1 + \sum_{j=1}^{d} a_j\big(x^{2g-4-2j} + x^{2j}\big)\right)$$

over the complement of the discriminant locus $\Delta \subset \mathbb{A}_{\mathbb{Q}}^d = \mathrm{Spec}(\mathbb{Q}[a_1, \dots, a_d])$. We will be particularly interested in the case $g = 4$ (with $d = 1$), given by

$$y^2 = x(x^4 - 1)(x^4 + 2a_1 x^2 + 1)$$

with $\Delta = -2^{40}(a_1^2 - 1)$. The fibers $\mathsf{C}_a^{(g)}$ for $a \in (\mathbb{A}^d \smallsetminus \Delta)(K)$ are nice curves of genus g with $Q_8 \hookrightarrow \mathrm{Aut}\,\mathsf{C}_a^{(g)}$ whenever $i = \sqrt{-1} \in K$; accordingly, their Jacobians $\mathsf{A}_a^{(g)} := \mathrm{Jac}\,\mathsf{C}_a^{(g)}$ have endomorphisms by the Lipschitz (quaternion) order

$$(1.2.3) \qquad \mathcal{O} := \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k \subset \left(\frac{-1, -1}{\mathbb{Q}}\right)$$

whenever $i \in K$.

The family $\mathsf{C}^{(g)}$ has many pleasant properties. We record (Proposition 3.2.4) the fact that this family is 'universal' in the sense that every hyperelliptic curve of genus g with a certain action (3.2.1) by Q_8 over K is isomorphic to a specialization $\mathsf{C}_a^{(g)}$ for some $a \in K^d$, up to quadratic twist. We also observe that, whenever a curve C has geometrically simple Jacobian and an action by Q_8, then it is hyperelliptic (Lemma 3.1.1). Finally, for $g = 4$, we show the family is universal in a second sense: the closure $\mathbb{P}^1$ of the base $\mathbb{A}^1 \smallsetminus \Delta$ is in fact the coarse space of a certain moduli space of abelian varieties of explicit PEL type (Theorem 7.2.2); it is striking that this Shimura variety admits such an explicit description.

With this family in hand, our main result—one that implies a precise form of Theorem 1.2.1—is as follows.

Theorem 1.2.4. *Let $g \geq 4$ be even. Then for a density 1 subset of points $a = (a_1, \dots, a_d) \in \mathbb{Q}^d$ ordered by height, the curve $\mathsf{C}_a^{(g)}$ over $\mathbb{Q}$ is nice of genus g and its Jacobian $A := \mathsf{A}_a^{(g)}$ satisfies all of the following:*

  (i) the geometric endomorphism ring is $\mathrm{End}\,A^{\mathrm{al}} = \mathcal{O}$;
  (ii) the endomorphism field of A is $\mathbb{Q}(\mathrm{End}\,A) = \mathbb{Q}(i)$;
  (iii) the connected monodromy field of A is $\mathbb{Q}(\varepsilon_A) = \mathbb{Q}(\zeta_8)$;
  (iv) A satisfies the Hodge, Tate, and Mumford–Tate conjectures; and
  (v) the Mumford–Tate group of A is a $\mathbb{Q}$-form of $\mathbb{G}_m \cdot \mathrm{SO}_g$.

Moreover, as $a \in \mathbb{Q}^d$ varies within this subset, the Jacobians $\mathsf{A}_a^{(g)}$ fall into infinitely many distinct $\mathbb{Q}^{\mathrm{al}}$-isogeny classes.

Theorem 1.2.4 relates to several recent papers.

- We complete a recent example of Zywina [Zyw20, Section 1.8]: we prove that for the Jacobian of the curve $y^2 = x(x^{20} + 7x^{18} - 7x^2 - 1)$ over $\mathbb{Q}$, the connected

monodromy field is $\mathbb{Q}(\zeta_8)$, as claimed. For details, see Example 5.2.2, with key input provided by the theorem of Banaszak–Cantoral-Farfán mentioned above.

- In upcoming work by Gajda–Hindry [HG23], they prove that Mumford–Tate holds for certain Jacobians of the form $\mathsf{A}_a^{(g)}$. Part (i) of Theorem 1.2.4 proves their conjecture that the geometric endomorphism ring of $\mathsf{A}_a^{(g)}$ is equal to $\mathcal{O}$ for generic values of a.
- Goodson [Goo22, §5.2.3] gives an example of an abelian variety of Fermat type where the Sato–Tate group (see Remark 2.2.6) becomes connected over a degree 2 extension of the endomorphism field. (However, her identification of this extension is only conjectural, based on computational evidence.)

The family (1.2.2) is indeed remarkable, but our approach also suggests a general strategy to understand the arithmetic of families of Jacobians of curves with large automorphism group.

1.3. **Context and discussion.** It remains a fundamental problem that certain abelian varieties carry exceptional classes in their (Betti or ℓ-adic étale) cohomology which are not as yet known to be linear combinations of algebraic classes [Mur84]. Such exceptional classes can only exist on abelian varieties of dimension $g \geq 4$. This can be explained quickly by combining Poincaré duality with the Lefschetz theorem on $(1,1)$ classes (or its ℓ-adic counterpart, due to Faltings): all Hodge classes in $H^2(A)$ are algebraic, hence by duality the same is true for classes in $H^{2g-2}(A)$, and this suffices to prove the Hodge conjecture for abelian varieties of dimension $g \leq 3$. Therefore, the first nontrivial case to consider is $H^4(A)$ with $\dim A = 4$, which is the topic of this paper.

By work of Murty [Mur84, Theorem 3.1], the following are equivalent for a simple abelian variety A:

- every algebraic class on any power of A is generated by divisor classes; and
- the Hodge and Lefschetz groups of A coincide (see Definition 2.2.1), and A is not of Type III in the sense of Albert.

One can show that all divisor classes are defined over the field $K(\mathrm{End}\, A)$: working in étale cohomology, we have that $H^2_{\text{ét}}(A^{\text{al}}, \mathbb{Q}_\ell)(1)$ is a direct summand of

$$(1.3.1) \qquad \left(H^1_{\text{ét}}(A^{\text{al}}, \mathbb{Q}_\ell) \otimes H^1_{\text{ét}}(A^{\text{al}}, \mathbb{Q}_\ell) \right)(1) = \mathrm{Hom}_{\mathbb{Q}_\ell}(V_\ell(A), V_\ell(A)),$$

see for example Tate [Tat66, p. 143]. It follows that any (divisor) class in $H^2_{\text{ét}}(A^{\text{al}}, \mathbb{Q}_\ell)(1)$ fixed by a finite-index subgroup $\mathrm{Gal}_L \leq \mathrm{Gal}_K$ lies in the subspace of Gal_L-fixed points

$$\mathrm{Hom}_{\mathbb{Q}_\ell}(V_\ell(A), V_\ell(A))^{\mathrm{Gal}_L} = \mathrm{End}(A_L) \otimes \mathbb{Q}_\ell,$$

hence it is fixed by $\mathrm{Gal}_{K(\mathrm{End}\, A)}$. Thus simple abelian varieties A with $K(\mathrm{End}\, A) \neq K(\varepsilon_A)$ should either have different Hodge and Lefschetz groups or they should be of Type III. The former seems challenging to construct explicitly, so in this paper we focus on the case of varieties of Type III. For these, Murty [Mur84, Theorem 3.2] shows the existence of Hodge classes in A^2 not generated by divisors. These classes underlie many of the constructions in this paper. We describe them geometrically (see section 5.2) as Weil classes for a quadratic

subfield of the endomorphism ring (Remark 5.3.4) and as a geometric incarnation of a certain determinant on a submodule of cohomology (Remark 5.3.3).

1.4. **Strategy of proof.** The overall strategy of the proof of our main result (Theorem 1.2.4) is first to give an essentially direct argument in genus $g = 4, 6$, spreading out from explicit computation on a specialization, then to argue by induction to treat the general case $g \geq 4$. Significant input is given by the aforementioned theorem of Banaszak–Cantoral-Farfán [BCF22] and the construction of cycles provided by Schoen [Sch88]. As an alternative to our inductive approach, it may be possible to establish some parts of our main result by studying the Galois action on the 2-torsion points (as in Zarhin [Zar00]).

We now make a few comments about the proof of part (iv) of Theorem 1.2.4. On one hand, it is known in many cases that every 'sufficiently general' member of a family of abelian varieties satisfies the Mumford–Tate conjecture. For example, Zywina [Zyw19] shows that, in a family of abelian varieties over a rational base, a density 1 subset of the fibres have the same ℓ-adic monodromy group as the generic fibre; this can often be used to show that a density 1 subset of the fibres satisfy the Mumford–Tate conjecture. We also adopt this approach. Noot [Noo95] (based on ideas of Serre) constructs families of abelian varieties whose Mumford–Tate group is contained in a given group G, and he shows that 'most' members of these families (meaning away from a thin set on the base) satisfy the Mumford–Tate conjecture. Moreover, the Mumford–Tate conjecture is known for certain abelian varieties of Type III [BGK10, Pin98, CF19].

On the other hand, there are not many known cases of the Hodge and Tate conjectures for abelian varieties of Type III—this is essentially due to the existence of exceptional classes. To prove these conjectures for the Jacobians studied in this paper, we rely on previous work of Schoen [Sch88] on abelian varieties with an automorphism. However, Schoen's result is not enough to establish part (iii) of Theorem 1.2.4, because we also need to know the minimal field of definition of these algebraic classes—we determine this field by writing down explicit cycles representing them.

1.5. **Organization.** After dispensing with preliminaries in section 2, in section 3 we present our family of hyperelliptic curves with automorphisms by Q_8. In section 4, we consider a specialization in $g = 4$ and prove parts of Theorem 1.2.4 in this special case. In section 5 and section 6, we proceed with the general case $g \geq 4$ by an inductive argument. We conclude in section 7 with some final remarks, including the interpretation in $g = 4$ as a moduli space of abelian varieties.

1.6. **Acknowledgements.**

2. Preliminaries

In this section, we quickly establish basic notation and setup.

2.1. **Notation.** For a vector space V over a field K we denote by GL_V the algebraic group given as a functor on K-algebras B by $B \mapsto \mathrm{Aut}(V \otimes_K B)$. We denote by $\mathrm{GL}(V)$ the group of points $\mathrm{GL}_V(K) = \mathrm{Aut}_K(V)$. We use similar notation when V is a free module over a ring R. When V is a $\mathbb{Q}$-vector space and F is any field of characteristic 0, we write V_F for $V \otimes_{\mathbb{Q}} F$. If G is an algebraic subgroup of GL_V, we similarly write G_F for $G \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,F$.

2.2. **Abelian varieties and associated groups.** Let $K \subset \mathbb{C}$ be a finitely generated subfield with algebraic closure $K^{\mathrm{al}} \subset \mathbb{C}$. (The choice of embedding $K \hookrightarrow \mathbb{C}$ does not affect any of the following constructions by work of Deligne [Del82, Theorem 2.11].) Let A be an abelian variety of dimension g defined over K, with $A^{\mathrm{al}} := A \times_K K^{\mathrm{al}}$ its base change to K^{al}. For an algebraic extension $K' \supseteq K$, we denote by $\mathrm{End}(A_{K'})$ the endomorphism ring of $A_{K'} := A \times_K K'$ (endomorphisms defined over K'), and $\mathrm{End}(A_{K'})_{\mathbb{Q}} := \mathrm{End}(A_{K'}) \otimes \mathbb{Q}$. In particular, $B := \mathrm{End}(A^{\mathrm{al}})_{\mathbb{Q}}$ is the geometric endomorphism algebra of A.

We briefly recall the definition of several algebraic groups naturally attached to A. For further general reference, see Banaszak–Kedlaya [BK15, §4]. A polarisation of A yields a symplectic bilinear form ψ on $V := H_1(A(\mathbb{C}), \mathbb{Q})$. There is also a natural action of $\mathrm{End}(A^{\mathrm{al}})_{\mathbb{Q}}$ on V preserving ψ, giving a homomorphism $B^\times \to \mathrm{Sp}_{V,\psi}(\mathbb{Q})$, where $\mathrm{Sp}_{V,\psi}$ is the algebraic subgroup of GL_V given by the condition of preserving ψ.

Definition 2.2.1. The Lefschetz group of A is $L(A) := C_{\mathrm{Sp}_{V,\psi}}(B^\times)^0$, the identity component of the centraliser of the image of $B^\times$ (as an algebraic group) in $\mathrm{Sp}_{V,\psi}$.

The Lefschetz group is a connected, reductive $\mathbb{Q}$-algebraic subgroup of $\mathrm{Sp}_{V,\psi}$. It was first introduced by Murty [Mur84, §2]. The group $L(A)$ is the identity component of the group denoted by $S(A)$ in Milne [Mil99].

The Lefschetz group can be meaningfully compared with another algebraic group attached to A: namely, its Hodge group, defined as follows. Recall that V is naturally a Hodge structure of weight -1, i.e., we have $V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$ with the property that $\overline{V^{-1,0}} = V^{0,-1}$. Let $\mathbb{S} := \mathrm{Res}_{\mathbb{C}|\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ be the Deligne torus. Then equivalently we have a Hodge cocharacter

$$(2.2.2) \qquad\qquad h : \mathbb{S} \to \mathrm{GL}_{V_{\mathbb{R}}}$$

where $z = (z_1, z_2) \in \mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$ acts as multiplication by z_1 in $V^{-1,0}$ and as multiplication by z_2 in $V^{0,-1}$ [Moo17, §2.1].

Definition 2.2.3. The Mumford–Tate group of A, denoted $\mathrm{MT}(A)$, is the smallest $\mathbb{Q}$-algebraic subgroup of GL_V such that h factors via $\mathrm{MT}(A)_{\mathbb{R}}$. The Hodge group of A, denoted $\mathrm{Hg}(A)$, is the identity component of the intersection $\mathrm{MT}(A) \cap \mathrm{Sp}_{V,\psi}$.

By construction, $\mathrm{MT}(A)$ is the almost-direct product inside GL_V of $\mathrm{Hg}(A)$ with $\mathbb{G}_m$, considered as the central torus of homotheties.

For a prime number ℓ, let $T_\ell = T_\ell(A) := \varprojlim_n A[\ell^n] \simeq H^1_{\text{ét}}(A^{\mathrm{al}}, \mathbb{Z}_\ell)^\vee$ be the ℓ-adic Tate module of A. The absolute Galois group $\mathrm{Gal}_K := \mathrm{Gal}(K^{\mathrm{al}} \,|\, K)$ acts naturally on the étale cohomology of A^{al} and therefore we have a continuous ℓ-adic representation

$$(2.2.4) \qquad\qquad \rho_{A,\ell} : \mathrm{Gal}_K \to \mathrm{GL}(T_\ell) < \mathrm{GL}(V_\ell),$$

where $V_\ell = V_\ell(A) := T_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

Definition 2.2.5. The ℓ-adic monodromy group of A, denoted $G_{A,\ell}$, is $\overline{\rho_{A,\ell}(\mathrm{Gal}_K)}^{\mathrm{Zar}} \leq \mathrm{GL}_{V_\ell}$, the Zariski closure of the image of the ℓ-adic representation.

We abbreviate $G_\ell = G_{A,\ell}$ when A is clear from context, and we write $G_\ell^0 = G_{A,\ell}^0$ for its identity component. We similarly define the special ℓ-adic monodromy group $G_{\ell,1} := G_\ell \cap \mathrm{SL}_{V_\ell}$.

Remark 2.2.6. We should also mention the Sato–Tate group of A, a maximal compact subgroup of $G_{\ell,1}(\mathbb{C})$ [Ser12, §8.3.3] (unique up to conjugation) obtained from a chosen embedding $\mathbb{Q}_\ell^{\mathrm{al}} \hookrightarrow \mathbb{C}$. Although the Sato–Tate group will not play a featured role here, our main result informs its description for fibers in our family.

The Lefschetz, Mumford–Tate, Hodge, and ℓ-adic monodromy groups of A can be compared to one another in the following precise sense. First, we may identify $V_{\mathbb{Q}_\ell} \simeq V_\ell(A)$ using Artin's comparison isomorphism in étale cohomology.

Theorem 2.2.7. *The following hold:*

    (a) *For all ℓ, we have $G_\ell^0 \leq \mathrm{MT}(A)_{\mathbb{Q}_\ell}$ and $G_{\ell,1}^0 \leq \mathrm{Hg}(A)_{\mathbb{Q}_\ell}$.*
    (b) *We have $\mathrm{Hg}(A) \leq L(A)$.*

Proof. For part (a), see Deligne [Del82, Proposition 2.9, Theorem 2.11]. More precisely, every Hodge cycle on A is absolutely Hodge [Del82, Theorem 2.11] and (the étale components of) all absolutely Hodge cycles are defined over a finite extension L of K [Del82, Proposition 2.9(b)]. Thus, the absolute Galois group of L acts trivially on all Hodge cycles, which are precisely the fixed points of the action of the Mumford–Tate group. By Galois theory, this implies that $G_{A_L,\ell} \leq \mathrm{MT}(A)_{\mathbb{Q}_\ell}$, and in particular $G_\ell^0 \leq G_{A_L,\ell}$ is contained in $\mathrm{MT}(A)_{\mathbb{Q}_\ell}$, which proves (a). For part (b), see Milne [Mil99, p.665, before Proposition 4.8] or Murty [Mur84, Section 2, before Lemma 2.1]. □

Theorem 2.2.7(a) leads naturally to the following well-known conjecture.

Conjecture 2.2.8 (Mumford–Tate)**.** *Let A be an abelian variety over a finitely generated field $K \subset \mathbb{C}$. Then for all prime numbers ℓ, the natural embedding $G_{A,\ell}^0 \hookrightarrow \mathrm{MT}(A)_{\mathbb{Q}_\ell}$ is an isomorphism.*

On the other hand, in general we cannot expect the Lefschetz and Hodge groups to agree. Accordingly, we make the following definition.

Definition 2.2.9. We say that A is fully of Lefschetz type if the following two conditions hold:

    (i) the inclusion $\mathrm{Hg}(A) \leq L(A)$ is an equality; and
    (ii) the Mumford–Tate Conjecture holds for A.

2.3. **Type III.** Recall that if A is simple, then A is of Type III in the Albert classification if its geometric endomorphism algebra $B = \operatorname{End}(A^{\mathrm{al}})_{\mathbb{Q}}$ is a totally definite quaternion algebra. The center of B is then a totally real number field $F := Z(B)$, and the relative dimension of A is

$$(2.3.1) \qquad \operatorname{reldim}(A) := \frac{\dim A}{2[F:\mathbb{Q}]} = \frac{g}{2e} \in \mathbb{Z}_{\geq 1}$$

where $e := [F:\mathbb{Q}]$, and in particular $g = \dim A$ is even.

Lemma 2.3.2. *Suppose A is simple of Type III, and suppose its geometric endomorphism algebra has center $Z(B) = \mathbb{Q}$. Then the Lefschetz group $L(A)$ has rank $g/2$ and dimension $(g^2 - g)/2$, and $L(A)$ is a $\mathbb{Q}$-form of SO_g.*

Proof. We refer to Milne [Mil99, Summary of Section 2]. Notice that the dimension and rank of an algebraic group only depend on its identity component, and our $L(A)$ is the identity component of Milne's group $S(A)$. ☐

Remark 2.3.3. Let A be a simple abelian variety of Type III defined over a number field K. By a theorem of Banaszak–Gajda–Krasoń [BGK10, Theorem 5.11], the Mumford–Tate conjecture holds for A if $\operatorname{reldim}(A)$ is odd and not in an explicit list of exceptions (see Lombardo [Lom16b, Remark 2.27] and Cantoral-Farfán [CF19, Corollary 1.11]).

The Mumford–Tate conjecture also holds for simple abelian fourfolds of Type III, by work of Moonen–Zarhin [MZ95, §6]. On the other hand, for infinitely many even values of g, the Mumford–Tate conjecture is not known to hold for a general, simple g-dimensional abelian variety of Type III.

The following result was part of the inspiration for the investigations in this paper.

Theorem 2.3.4. *Let A be a simple abelian variety over K of Type III, fully of Lefschetz type. Then $[K(\varepsilon_A) : K(\operatorname{End} A)] \leq 2$.*

Proof. See Banaszak–Cantoral-Farfán [BCF22, Corollary 3.7, Corollary 3.8]. ☐

Remark 2.3.5. Theorem 2.3.4 can be extended to abelian varieties isogenous to a product of simple abelian varieties of Type III.

2.4. **Connected monodromy field.** The identity component $G^0_{A,\ell}$ of the ℓ-adic monodromy group is invariant under finite extensions of the ground field K. Moreover, as we observed in the introduction (1.1.3), there is a minimal extension $K(\varepsilon_A)$ of K over which all ℓ-adic monodromy groups become connected, and we have canonical group isomorphisms

$$(2.4.1) \qquad \operatorname{Gal}(K(\varepsilon_A) \,|\, K) \simeq \pi_0(G_{A,\ell})$$

for each prime ℓ.

We now describe the field $K(\varepsilon_A)$ in terms of the geometry of A and of its algebraic cycles. Recall that the cycle class map of étale cohomology attaches to each codimension d cycle Z on A^{al} a class $[Z]$ in $H^{2d}_{\mathrm{\acute{e}t}}(A^{\mathrm{al}}, \mathbb{Q}_\ell(d))$, invariant under a subgroup of Gal_K of finite index. The (minimal) field of definition of $\delta \in H^{2d}_{\mathrm{\acute{e}t}}(A^{\mathrm{al}}, \mathbb{Q}_\ell(d))$ is the fixed field of the finite-index subgroup of Gal_K that fixes δ.

In order to better understand the Galois action on $H^{2d}_{\text{ét}}(A^{\text{al}}, \mathbb{Q}_\ell(d))$, recall that, for all $i \geq 0$, we have an isomorphism

$$(2.4.2) \qquad H^i_{\text{ét}}(A^{\text{al}}, \mathbb{Q}_\ell) \simeq \bigwedge{}^i H^1_{\text{ét}}(A^{\text{al}}, \mathbb{Q}_\ell)$$

as Gal_K-representations.

Proposition 2.4.3. *The field of definition of $\delta \in H^{2d}_{\text{ét}}(A^{\text{al}}, \mathbb{Q}_\ell(d))$ is contained in $K(\varepsilon_A)$.*

Proof. Let L be the field of definition of δ. By definition, δ is invariant under the action of Gal_L hence also under the action of $G_{A_L,\ell}$ (via the $2d$-th exterior power of its defining representation, tensored with $\mathbb{Q}_\ell(d)$). In particular, it is invariant under

$$G^0_{A_L,\ell} = G^0_{A_{K(\varepsilon_A)},\ell} = G_{A_{K(\varepsilon_A)},\ell}.$$

Thus $\text{Gal}_{K(\varepsilon_A)}$ acts trivially on δ, hence $L \subseteq K(\varepsilon_A)$. $\qquad\square$

From the preceding proposition, it is not hard to characterise the field $K(\varepsilon_A)$ in geometric terms, at least under the assumption of the Tate conjecture.

Corollary 2.4.4. *Let L be the minimal extension of K such that, for all $k \in \mathbb{Z}_{\geq 1}$ and $d \in \mathbb{Z}_{\geq 0}$, and for all codimension d cycles Z on A^k, the field of definition of $[Z]$ is contained in L. Then the following statements hold:*

(a) $L \subseteq K(\varepsilon_A)$.
(b) *If the Tate conjecture for algebraic cycles holds for A^r for all $r \geq 1$, then $L \supseteq K(\varepsilon_A)$ (and hence $L = K(\varepsilon_A)$).*

Proof. Part (a) follows from Proposition 2.4.3. Conversely, we now show that $K(\varepsilon_A) \subseteq L$, or equivalently, that $G_{A_L,\ell}$ is connected for all ℓ, assuming the Tate conjecture for algebraic cycles on all powers of A.

For simplicity of notation, set $G := G_{A_L,\ell}$ and assume for purposes of contradiction that G is not connected. Then the finite group G/G^0 admits a faithful representation in some tensor construction over $V := H^1(A^{\text{al}}, \mathbb{Q}_\ell)$. Explicitly, by a theorem of Chevalley [Del82, Proposition 3.1(c)], there exists a finite collection of pairs (m_i, n_i) such that the subspace

$$(2.4.5) \qquad W := \left\{ x \in \bigoplus_i V^{\otimes m_i} \otimes (V^*)^{\otimes n_i} : g \cdot x = x \text{ for all } g \in G^0 \right\}$$

has the property that the subgroup of G fixing W pointwise is G^0. Equivalently, as stated, G acts on W via G/G^0, and the induced map $G/G^0 \to \text{GL}_W$ is injective. Now observe that, due to the existence of the G-equivariant bilinear form coming from a polarization, we have $V^\vee \simeq V(-1)$, so that W as above is contained in a direct sum of spaces of the form $V^{\otimes n}(-d)$.

Next, we notice that any $V^{\otimes n}$ occurs in the cohomology of some power of A, because it is a direct summand of $\bigwedge^n(V^{\oplus n}) \simeq H^n((A^{\text{al}})^n, \mathbb{Q}_\ell)$. The kernel of the natural map

$$\text{Gal}_L \to G(\mathbb{Q}_\ell) \to (G/G^0)(\mathbb{Q}_\ell)$$

defines a finite extension L' of L with the property that $\text{Gal}_{L'}$ acts trivially on W. By the Tate conjecture on algebraic cycles for A^r, this implies that W is spanned by classes of

algebraic cycles on certain powers of A. On the other hand, Gal_L does not act trivially on W, since its image in $G(\mathbb{Q}_\ell)$ (hence in $(G/G^0)(\mathbb{Q}_\ell)$) is dense. This contradicts the very definition of L and finishes the proof. $\qquad\square$

Remark 2.4.6. This also gives another proof of the containment $K(\mathrm{End}\,A) \subseteq K(\varepsilon_A)$ (1.1.5), since the graphs of endomorphisms are in particular algebraic cycles on A^2 (cf. (1.3.1)).

2.5. **Connected monodromy field: type III.**

For the remainder of this section, suppose that A is of type III, fully of Lefschetz type, with $Z(\mathrm{End}(A^{\mathrm{al}})_\mathbb{Q}) = \mathbb{Q}$. In particular, $g = \dim A$ is even. Then Theorem 2.3.4 shows that $[K(\varepsilon_A) : K(\mathrm{End}\,A)] \leq 2$. We prove some properties of the extension $K(\varepsilon_A) \supseteq K(\mathrm{End}\,A)$. Since we are only interested in this extension, we will suppose that $K(\mathrm{End}\,A) = K$.

Recall from Banaszak–Gajda–Krasoń [BGK10] that for all sufficiently large primes ℓ, there is a $\mathbb{Q}_\ell$-vector space $W_\ell(A)$ equipped with a Gal_K-action together with a Gal_K-equivariant isomorphism

$$(2.5.1) \qquad\qquad V_\ell(A) \simeq W_\ell(A)^{\oplus 2};$$

moreover, there is a nondegenerate, symmetric bilinear form

$$(2.5.2) \qquad\qquad \psi_\ell \colon W_\ell(A) \times W_\ell(A) \to \mathbb{Q}_\ell$$

that is also Gal_K-equivariant. We have $\dim W_\ell(A) = (\dim V_\ell(A))/2 = g$. Let $\mathrm{GO}_{W_\ell(A),\psi_\ell}$ be the group of similitudes of ψ_ℓ, also known as the conformal group of ψ_ℓ, with similitude character

$$(2.5.3) \qquad\qquad \chi_\ell \colon \mathrm{GO}_{W_\ell(A),\psi_\ell} \to \mathbb{Q}_\ell^\times$$

so that

$$\psi_\ell(\gamma x, \gamma y) = \chi_\ell(\gamma)\psi_\ell(x,y)$$

and $\chi_\ell(\gamma)^g = \det(\gamma)^2$ for all $\gamma \in \mathrm{GO}_{W_\ell(A),\psi_\ell}$ and $x,y \in W_\ell(A)$. The restriction of χ_ℓ to $G_{A,\ell}$ is the cyclotomic character.

The algebraic monodromy group is a subgroup of $\mathrm{GO}_{W_\ell(A),\psi_\ell}$. Moreover, as A is fully of Lefschetz type, for every ℓ the connected component $G_{A,\ell}^0$ is the group of *oriented* similitudes of ψ_ℓ, denoted

$$\mathrm{GO}_{W_\ell(A),\psi_\ell}^0 = \mathrm{GO}_{W_\ell(A),\psi_\ell}^+ = \mathrm{GSO}_{W_\ell(A),\psi_\ell},$$

defined as the kernel of the character

$$(2.5.4) \qquad \begin{aligned} \omega_{A,\ell} = \omega_\ell \colon \mathrm{GO}_{W_\ell(A),\psi_\ell} &\to \{\pm 1\} \\ \omega_\ell(\gamma) &= \det(\gamma)\chi_\ell(\gamma)^{-g/2}. \end{aligned}$$

(The character ω_ℓ is trivial on scalars, and upon restriction to $\mathrm{O}_{W_\ell(A),\psi_\ell}$ it agrees with $\det$; of course $\mathrm{SO}_{W_\ell(A),\psi_\ell}$ is the kernel of $\det$ inside $\mathrm{O}_{W_\ell(A),\psi_\ell}$, so the description follows.)

Restricting this character to Gal_K gives the following. Let $\mathfrak{p}$ be a prime of good reduction for A with absolute norm $q := \mathrm{Nm}(\mathfrak{p})$, and let $\mathrm{Frob}_\mathfrak{p} \in \mathrm{Gal}_K$ be a $\mathfrak{p}$-Frobenius element. Then

$$(2.5.5) \qquad \det(1 - \mathrm{Frob}_\mathfrak{p}\,T \,|\, V_\ell(A)) = \det(1 - \mathrm{Frob}_\mathfrak{p}\,T \,|\, W_\ell(A))^2$$

and $\det(\rho_{A,\ell}(\mathrm{Frob}_{\mathfrak{p}})) = q^g$, so $\det(\mathrm{Frob}_{\mathfrak{p}} \,|\, W_\ell(A)) = \pm q^{g/2} = \omega_\ell(\mathrm{Frob}_{\mathfrak{p}})\chi_\ell(\mathrm{Frob}_{\mathfrak{p}})^{g/2}$. In particular, the character $\omega = \omega_A := \omega_{A,\ell} \circ \rho_{A,\ell}$ of Gal_K is independent of ℓ.

Lemma 2.5.6. *We have $\rho_{A,\ell}(\mathrm{Frob}_{\mathfrak{p}}) \in G^0_{A,\ell}(\mathbb{Q}_\ell)$ if and only if $\det(\mathrm{Frob}_{\mathfrak{p}} \,|\, W_\ell(A)) = q^{g/2}$.*

Proof. We have $\gamma \in G^0_{A,\ell}(\mathbb{Q}_\ell)$ if and only if γ, restricted to one copy of $W_\ell(A)$, is in $\mathrm{GSO}_{W_\ell(A),\psi_\ell}$. $\qquad\square$

Lemma 2.5.7. *ω is unramified at primes of (good or) semistable reduction of A.*

Proof. Let ℓ be prime. Since $\rho_{A,\ell}$ is unramified outside the set consisting of the primes above ℓ and the primes of bad reduction of A, so too is ω. By independence of ℓ, we conclude ω is only ramified at primes of bad reduction.

Let now $\mathfrak{p}$ be a (nonzero) prime of the ring of integers of K at which A has bad but semistable reduction. Let $\ell > 2$ be coprime to $\mathfrak{p}$. By the Galois criterion for semistable reduction, the inertia group at $\mathfrak{p}$ acts on $T_\ell(A)$ via unipotent matrices of echelon 2. Both the determinant and the multiplier of such a matrix are equal to 1, so by definition ω is trivial on the inertia group at $\mathfrak{p}$, as desired. $\qquad\square$

3. CURVES WITH Q_8 AUTOMORPHISMS

In this section, we present a certain universal family of curves with automorphisms by the quaternion group Q_8 of order 8.

3.1. **Setup.** We begin with a brief setup. Throughout, we may relax our hypothesis on the field K, asking only that $\mathrm{char}\,K \neq 2$. For S a variety over K, a family of nice curves of genus g over S is a flat morphism $\mathsf{C} \to S$ whose fibres are nice (smooth, projective, geometrically connected) curves of genus g.

To frame our family, we begin with the following lemma.

Lemma 3.1.1. *Let C be a nice curve over K of genus $g \geq 2$ and let $G \leq \mathrm{Aut}(C^{\mathrm{al}})$ be a subgroup isomorphic to Q_8, stable under Gal_K. Suppose further that $\mathrm{Jac}(C)$ is simple. Then C is hyperelliptic and g is even.*

Proof. Let $\iota \in G \simeq Q_8$ be the nontrivial central element. Then ι is characteristic in G so defined over K. Then the morphism $C \to C' := C/\langle\iota\rangle$ is a map of curves over K of degree 2. Let g' be the genus of C'. If $g' \geq 1$, then $\mathrm{Jac}(C')$ is an isogeny factor of $\mathrm{Jac}(C)$; by Riemann–Hurwitz we have $g' < g$, so the cofactor is nontrivial, which contradicts that $\mathrm{Jac}(C)$ is simple. So $g' = 0$, the curve C is geometrically hyperelliptic, and ι is the hyperelliptic involution.

To prove that G is in fact hyperelliptic (i.e., we have $C' \simeq \mathbb{P}^1_K$), we look at the action of G on the rational 2-adic Tate module $V_2(A)$. Since G is Galois stable, $V_2(A)$ has the structure of a $\mathbb{Q}_2[G]$-module compatible with the Gal_K-action. In the decomposition of $V_2(A)$ into irreducibles under G, in the previous paragraph we proved that the trivial representations does not occur; in fact, there can also be no one-dimensional characters because $\langle\iota\rangle$ acts trivially on such characters. But we have the Wedderburn decomposition

$\mathbb{Q}_2[Q_8] \simeq \mathbb{Q}_2^4 \times \left(\dfrac{-1,-1}{\mathbb{Q}_2}\right)$, and the quaternion algebra $\left(\dfrac{-1,-1}{\mathbb{Q}_2}\right)$ is a division algebra. Thus $V_2(A)$ is a direct sum of copies of this remaining irreducible, and by dimensions we get $2g \equiv 0 \pmod 4$. Thus g is even. But then it is well-known that geometrically hyperelliptic implies hyperelliptic: the image C' of C under the canonical embedding has odd degree $g-1$, so C' has an odd degree point, and from a multiple of the canonical class we obtain a divisor on C' over K of degree 1 and hence $C'(K) \neq \emptyset$, whence $C' \simeq \mathbb{P}^1_K$. $\square$

Remark 3.1.2. The classification of hyperelliptic curves with Q_8 action is classical. For actions on non-hyperelliptic curves, see Kimura [Kim93].

Lemma 3.1.3. *Let C be a hyperelliptic curve over K with $Q_8 \hookrightarrow \mathrm{Aut}(C)$ defined over K. Then $\left(\dfrac{-1,-1}{K}\right) \simeq \mathrm{M}_2(K)$.*

Proof. The Q_8 action on C gives an action by $Q_8/\{\pm 1\} \simeq C_2 \times C_2$ on $\mathbb{P}^1_K$, i.e., we have a homomorphism $Q_8 \to \mathrm{PGL}_2(K)$ with kernel $\{\pm 1\}$. This projective representation lifts (by a direct calculation [LLG19, Corollary 2.5]) to $Q_8 \hookrightarrow \mathrm{GL}_2(K)$, hence to a surjective K-algebra homomorphism $K[Q_8] \to \mathrm{M}_2(K)$. Again by Wedderburn, we have $K[Q_8] \simeq K^4 \times \left(\dfrac{-1,-1}{K}\right)$ so we must have $\mathrm{M}_2(K) \simeq \left(\dfrac{-1,-1}{K}\right)$. $\square$

Corollary 3.1.4. *There is no hyperelliptic curve over $\mathbb{Q}$ with Q_8 action defined over $\mathbb{Q}$.*

Proof. Immediate from Lemma 3.1.3, since $\left(\dfrac{-1,-1}{\mathbb{Q}}\right)$ is a division algebra. $\square$

3.2. **Definition of family.** By the results in the previous section, in order to define a family of hyperelliptic curves of even genus over $\mathbb{Q}$ with an action by Q_8 over an extension of $\mathbb{Q}$, we will have to choose a splitting field for the quaternion algebra $\left(\dfrac{-1,-1}{\mathbb{Q}}\right)$.

We choose $\mathbb{Q}(i)$ as a splitting field for $\left(\dfrac{-1,-1}{\mathbb{Q}}\right)$ and the homomorphism

$$\rho \colon Q_8 \to \mathrm{GL}_2(\mathbb{Q}(i))$$

(3.2.1)
$$i, j \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

This choice is convenient for our calculations; any other such map over $\mathbb{Q}(i)$ will be conjugate by an element of $\mathrm{GL}_2(\mathbb{Q}(i))$, giving an isomorphic family below. Another choice of splitting field would correspondingly twist this family.

For Q_8 to act on a hyperelliptic curve $y^2 = f(x)$ with induced action (3.2.1) on the underlying $\mathbb{P}^1$, we need

(3.2.2)
$$f(-x) = -f(x) \quad \text{and} \quad f(1/x) = -f(x)/x^{2g+2}.$$

A calculation of invariants then leads to the following family.

Let $g \geq 2$ be an even integer and let $d := g/2 - 1$. We first consider the (relatively projective) curve over $\mathbb{A}_{\mathbb{Q}}^{d+1}$ obtained as the completion of the relative affine curve with hyperelliptic equation

$$(3.2.3) \qquad y^2 = x(x^4 - 1) \left(\sum_{j=0}^{d} a_j (x^{2g-4-2j} + x^{2j}) \right).$$

Proposition 3.2.4. *Let X be a hyperelliptic curve of even genus g over K with an action by Q_8 defined over $K(i)$ as in (3.2.1). Then X arises as a specialization of the family defined in (3.2.3).*

Proof. The first condition in (3.2.2) yields that every monomial in $f(x)$ has odd degree, hence in particular $f(0) = 0$. The two conditions in (3.2.2) taken together also imply that $f(x)$ vanishes for $x = \pm 1, \pm i$. Thus we can factor $f(x) = x(x^4 - 1)g(x)$, and (3.2.2) then yields that $g(x)$ is a self-reciprocal polynomial in x^2, so that $f(x)$ is as in the right-hand side of (3.2.3). $\qquad \square$

Up to twist, in the family (3.2.3) we may take $a_0 = 1$. So for the remainder of the paper, we work with the following setup.

Setup 3.2.5. Let $g \geq 4$ be even and let $d := g/2 - 1$, so $g = 2d + 2$. Let

$$f(x) = f^{(g)}(x) := x(x^4 - 1) \left(x^{2g-4} + 1 + \sum_{j=1}^{d} a_j(x^{2g-4-2j} + x^{2j}) \right) \in \mathbb{Q}[a_1, \ldots, a_d][x]$$

and let $\Delta = \Delta^{(g)}(a_1, \ldots, a_d) \in \mathbb{Q}[a_1, \ldots, a_d]$ be the discriminant of $f(x)$. Define the family of nice curves

$$(3.2.6) \qquad \mathsf{C} = \mathsf{C}^{(g)} \colon y^2 = f(x)$$

over U, where $U := \operatorname{Spec} \mathbb{Q}[a_1, \ldots, a_d, \Delta^{-1}] \subseteq \mathbb{A}_{\mathbb{Q}}^d$ is the complement of the discriminant Δ. Let $\mathsf{A}^{(g)} \to U$ be the abelian scheme given by the relative Jacobian of $\mathsf{C}^{(g)}$.

Let $C = C^{(g)}$ be the generic fiber of C over $\mathbb{Q}(a) = \mathbb{Q}(a_1, \ldots, a_d) = \mathbb{Q}(U)$ and $A = A^{(g)}$ its Jacobian.

For every field extension $K \supseteq \mathbb{Q}$ and every $a = (a_1, \ldots, a_d) \in U(K)$, we denote by C_a the fiber of C over a and by A_a the Jacobian of C_a (over K).

Lemma 3.2.7. *The subscheme $U \subseteq \mathbb{A}_{\mathbb{Q}}^d$ is nonempty and open.*

Proof. The polynomial Δ does not vanish identically: indeed, specializing at $a = (0, \ldots, 0)$ gives $x(x^4 - 1)(x^{2g-4} + 1)$ which, since $g \geq 2$ is even, has no repeated roots. $\qquad \square$

Remark 3.2.8. We will not need it in what follows, but the family can be extended over the base ring $\mathbb{Z}[1/2]$ in place of $\mathbb{Q}$.

We will make extensive use of the Q_8-automorphisms of $\mathsf{C}_{U_{\mathbb{Q}(i)}}$ generated by

$$(3.2.9) \qquad \alpha(x, y) = (-x, iy), \quad \beta(x, y) = \left(\frac{1}{x}, \frac{iy}{x^{g+1}} \right)$$

satisfying $\alpha^2 = \beta^2 = \iota$ and $\beta\alpha = \iota\alpha\beta$ where $\iota(x, y) = (x, -y)$ is the hyperelliptic involution. For each $a \in \mathbb{Q}^d$, the action of the automorphisms α, β over $\mathbb{Q}(i)$ on the subspace

$$\mathbb{Q}(i)\, \frac{\mathrm{d}x}{y} \oplus \mathbb{Q}(i)\, x^{g-1}\, \frac{\mathrm{d}x}{y} \subseteq H^0(\mathsf{C}_a^{(g)}, \Omega^1)$$

of regular differentials of $\mathsf{C}_a^{(g)}$ is given by the representation of (3.2.1). The automorphisms α, β generate a subring of endomorphisms of $\mathsf{A}_a = \operatorname{Jac} \mathsf{C}_a$ isomorphic to the Lipschitz order

$$(3.2.10) \qquad\qquad \mathcal{O} := \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}ij \subset \left(\frac{-1, -1}{\mathbb{Q}}\right)$$

under the map $i, j \mapsto \alpha^*, \beta^*$ and defined over $\mathbb{Q}(i)$.

The essential task of the rest of the paper will be to investigate the Jacobians A_a for $a \in \mathbb{Q}^d$, in particular their endomorphism field and connected monodromy field.

Remark 3.2.11. We could also consider $g = 2$ ($d = 0$) in Setup 3.2.5: the base is a single point and the curve is

$$y^2 = x(x^4 - 1).$$

This well-studied curve has geometric automorphism group $\operatorname{GL}_2(\mathbb{F}_3)$ (a group of order 48), and its Jacobian is isogenous over $\mathbb{Q}(i)$ to the square of an elliptic curve with complex multiplication by $\mathbb{Q}(i)$.

3.3. **A base change.** It will be convenient to consider a cover of the base of our family where the defining hyperelliptic polynomial factors (labelling 2-torsion), defined as follows. Consider the (dominant, quasi-finite) morphism

$$\varphi : \operatorname{Spec} \mathbb{Q}[b_1, b_1^{-1}, (b_1^4 - 1)^{-1}, \ldots, b_d, b_d^{-1}, (b_d^4 - 1)^{-1}] \to \operatorname{Spec} \mathbb{Q}[a_1, \ldots, a_d]$$

described functorially on points as follows: given a $\mathbb{Q}$-algebra R and a point $(b_1, \ldots, b_d) \in (R^\times)^d$, we consider the polynomial

$$(3.3.1) \qquad\qquad x(x^4 - 1) \prod_{j=1}^{d} (x^2 - b_j^2) \left(x^2 - \frac{1}{b_j^2}\right) \in R[x]$$

expand out and equate coefficients with the polynomial $f^{(g)}(x)$ in Setup 3.2.5, and then define $\varphi(b_1, \ldots, b_d) = (a_1, \ldots, a_d)$.

Example 3.3.2. For $g = 4$, we have $-2a_1 = b_1^2 + 1/b_1^2$.

The pullback $\varphi^*(\Delta)$ of the discriminant Δ from Setup 3.2.5 yields a nonempty open $V := \operatorname{Spec} \mathbb{Q}[b_1, \ldots, b_d, \varphi^*(\Delta)^{-1}]$ and the restriction $\varphi \colon V \to U$ is again dominant and surjective. By construction, we have a Cartesian diagram

$$(3.3.3)$$

$$\begin{array}{ccc} \mathsf{D}^{(g)} & \longrightarrow & \mathsf{C}^{(g)} \\ \downarrow & & \downarrow \\ V & \xrightarrow{\ \varphi\ } & U \end{array}$$

where $\mathsf{D}^{(g)} \to V$ is the family of nice hyperelliptic curves with affine equation

$$(3.3.4) \qquad \mathsf{D}^{(g)} \colon y^2 = x(x^4 - 1) \prod_{j=1}^{d} (x^2 - b_j^2) \left(x^2 - \frac{1}{b_j^2} \right).$$

Let $\mathbb{Q}(b) := \mathbb{Q}(V) = \mathbb{Q}(b_1, \ldots, b_d)$ be the function field of V. The map φ induces an injection $\varphi^*(\mathbb{Q}(a)) \hookrightarrow \mathbb{Q}(b)$ that we will simply write as an inclusion.

Adjoining the roots of $f(x)$ gives 2-torsion classes in the Jacobian, as follows. In view of (3.3.4), let

$$(3.3.5) \qquad S := \{0, \pm 1, \pm i\} \cup \{\pm b_j, \pm 1/b_j\}_{j=1,\ldots,r} \subset \mathbb{Q}(i)[b_1, 1/b_1, \ldots, b_d, 1/b_d]$$

so that $(x, 0) \in \mathsf{D}^{(g)}(V_{Q(i)})$ for $x \in S$, together with the unique point ∞ at infinity, are the Weierstrass points. The points with $x \in \{0, \pm 1, \pm i\}$ are distinguished from others by their stabilizers under the Q_8-automorphism group (3.2.9):

$$(3.3.6) \qquad \mathrm{Stab}_{Q_8}(x, 0) = \begin{cases} \langle \alpha \rangle, & \text{if } x = 0; \\ \langle \beta \rangle, & \text{if } x = \pm 1; \\ \langle \alpha\beta \rangle, & \text{if } x = \pm i; \\ \langle \iota \rangle, & \text{otherwise.} \end{cases}$$

(The point ∞ like $(0, 0)$ has stabilizer $\langle \alpha \rangle$.) In particular, the points with $x \in \{0, \pm 1, \pm i\}$ have Q_8-stabilizers of order 4 (the others have order 2).

For $x \in S$, let e_x be the divisor class $[(x, 0) - \infty]$ on $\mathsf{B}^{(g)}$. It is well-known that $\mathsf{A}^{(g)}[2]$ has a basis consisting of the divisor classes e_x for $x \in R \smallsetminus \{0\}$ and $\sum_{x \in R} e_x = 0$. In particular, $\mathsf{B}^{(g)}$ has full level 2 structure over $V_{\mathbb{Q}(i)}$.

The action of the Q_8-automorphism group on $\mathsf{A}^{(g)}[2]$ is similar: for $x \in R$, we have

$$(3.3.7) \qquad \alpha(e_x) = e_{-x}$$

and since $-e_0 = e_0$ is 2-torsion, we have

$$(3.3.8) \qquad \beta(e_x) = \begin{cases} e_{1/x} + e_0, & \text{if } x \neq 0; \\ e_0, & \text{if } x = 0. \end{cases}$$

Remark 3.3.9. We could also write $f(x)/x^{g-2} = x(x^4 - 1)h(x^2 + 1/x^2)$ where $h(y) = y^d + \sum_{j=1}^{d} a_j' y^{d-j}$ with $a_j' \in a_j + \mathbb{Q}[a_1, \ldots, a_{j-1}]$; by universality, $h(y)$ has Galois group S_d, so by back substitution $f(x)$ has generic Galois group $D_2 \wr S_d = D_2^d \rtimes S_d$ of order $4^d d!$ (over $\mathbb{Q}(i)(a_1, \ldots, a_d)$), where $D_2 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is the dihedral group of order 4.

Remark 3.3.10. One could similarly consider the intermediate base change with defining equation $y^2 = x(x^4 - 1) \prod_{i=1}^{d} (x^4 + c_i x^2 + 1)$, which corresponds to a (partial) level-2 structure on the Jacobian.

3.4. **Moduli.** Let $\mathcal{M}_g$ be the moduli space of curves of genus g and let M_g be its coarse space. There is a morphism $U \to \mathcal{M}_g \to M_g$ (with image in the hyperelliptic locus). The next lemma describes this map (generically).

Lemma 3.4.1. *The following statements hold.*

(a) *The geometric automorphism group of the generic fiber C is Q_8, generated by α, β in* (3.2.9) *and defined over $\mathbb{Q}(a)(i)$.*

(b) *The map $U \times_{\mathbb{Q}} \mathbb{Q}(i) \to M_g$ is generically Galois over its image, with Galois group S_3; in particular, the map has degree 6 and the image has dimension d.*

(c) *For $g = 4$ and $a \in \mathbb{C} \smallsetminus \{\pm 1\}$, we have $\operatorname{Aut} C_a \simeq Q_8$ except for the roots of*

$$x^{24} - 20x^{20} - 475x^{16} + 475x^8 + 20x^4 - 1;$$

among the remaining values, the curves C_a and $C_{a'}$ are isomorphic if and only if $a' \in \{\pm a, \pm(2a + 6)/(a - 1), \pm(2a - 6)/(a + 1)\}$.

For the proof of Theorem 1.2.4, we will only need the weaker statement in (b) that the map is finite-to-one, and we will not make use of part (c)—but we find the additional explicitness to be quite agreeable.

Proof. Moduli of hyperelliptic curves (in terms of binary forms) are well understood, and automorphism groups of hyperelliptic curves have been completely and explicitly classified (for characteristic zero, see Brandt–Stichtenoth [BS86] for the determination and Shaska [Sha13] for explicit moduli). From these results, one can deduce the lemma—we take a direct, self-contained approach.

First, part (a). There is a natural map

$$(3.4.2) \qquad \operatorname{Aut} C^{\mathrm{al}} \to \operatorname{Aut}(C^{\mathrm{al}}/\langle\iota\rangle) \simeq \operatorname{Aut}(\mathbb{P}^1_{\mathbb{Q}(a)^{\mathrm{al}}}) \simeq \operatorname{PGL}_2(\mathbb{Q}(a)^{\mathrm{al}})$$

with kernel $\langle\iota\rangle$; let G be its image, the reduced, geometric, generic automorphism group of C. Since any (geometric) automorphism of C stabilizes the set of Weierstrass points, any $\gamma \in G$ permutes the set $S \cup \{\infty\}$ of their x-coordinates in (3.3.5).

For the case $g = 4$, we verify (using a simple loop in Magma [BCP97]) that the only linear fractional transformations that stabilize the set $S^{(4)} := \{0, \infty, \pm 1, \pm i, \pm b_1, \pm 1/b_1\}$ are the maps $\gamma(x) = \pm x, \pm 1/x$ arising from the image $V_4 \simeq C_2^2$ of Q_8. More precisely, we check that for any three distinct $z_1, z_2, z_3 \in S^{(4)}$, the unique linear transformation

$$(3.4.3) \qquad \gamma(x) = \left(\frac{z_2 - z_3}{z_2 - z_1}\right) \frac{x - z_1}{x - z_3}$$

with $\gamma(z_1) = 0$, $\gamma(z_2) = 1$, and $\gamma(z_3) = \infty$, has $\gamma(S^{(4)}) \subseteq S^{(4)}$ if and only if $\gamma(x) = \pm x, \pm 1/x$ if and only if γ stabilizes $\{0, \pm 1, \infty\}$.

The general case $g \geq 4$ follows by specialization, as follows. Given γ with $\gamma(S) = S$, we have $\gamma(z_1) = 0$, $\gamma(z_2) = 1$, and $\gamma(z_3) = \infty$ with $z_1, z_2, z_3 \in S$ distinct. Assume for purposes of contradiction that $\gamma(x) \neq \pm x, \pm 1/x$. Then the set $\{0, \pm 1, \infty\}$ is not stabilized by γ. We specialize as follows:

- If $z_1 \in S^{(4)}$, we continue to the next step; otherwise $z_1 = \pm b_j, \pm 1/b_j$ for some $j > 1$, and we specialize $b_j = \pm b_1, \pm 1/b_1$ not equal to z_2 or z_3.

- If the specialization of z_2 lies in $S^{(4)}$, we continue; otherwise, $z_2 = \pm b_j, \pm 1/b_j$ with $j > 1$ and we specialize b_j to an element of $\{\pm b_1, \pm 1/b_1\}$ not equal to the specialization of z_1 or z_3.
- We repeat the previous step with z_3: if its specialization lies in $S^{(4)}$ we continue, else we specialize b_j to $\{\pm b_1, \pm 1/b_1\}$ not the specialization of z_1 or z_2.
- We specialize the remaining variables to b_1.

This yields a specialization homomorphism $\mathbb{Q}[b_1, \ldots, b_d, b_1^{-1}, \ldots, b_d^{-1}] \to \mathbb{Q}[b_1, b_1^{-1}]$, restricting to a map $S \to S^{(4)}$, such that $\{z_1, z_2, z_3\}$ restricts to a subset which is not contained in $\{0, \pm 1, \infty\}$. Specializing γ, we obtain a linear fractional transformation which stabilizes $S^{(4)}$, a contradiction.

For (b), from part (a) there exists a nonempty Zariski open subset $U' \subseteq U$ such that the family of nice curves $C_{U'}$ has automorphism group Q_8 over $U' \times_{\mathbb{Q}} \operatorname{Spec} \mathbb{Q}(i)$. We examine the fibers of the map $U' \to M_g$. For a $\mathbb{Q}(i)$-algebra R, let $a, a' \in U'(R)$ be such that $\varphi\colon \mathsf{C}_a \simeq \mathsf{C}_{a'}$. Then φ commutes with the hyperelliptic involution, so induces an isomorphism $\overline{\varphi}\colon \mathbb{P}^1 \to \mathbb{P}^1$ mapping the branch locus of C_a to that of $\mathsf{C}_{a'}$. But as we already observed in (3.3.6), the points with x-coordinates $\{0, \pm 1, \infty, \pm i\}$ have stabilizers of order 4, whereas the other points have stabilizer of order 2. Thus $\overline{\varphi}$ permutes $\{0, \pm 1, \infty, \pm i\}$. The subgroup of linear fractional transformations with this property form a group isomorphic to S_4 (naturally a subgroup of S_6) defined over $\mathbb{Q}(i)$ generated by $x \mapsto ix, 1/x, (x-1)/(x+1)$. The stabilizer of C_a is $Q_8/\langle \iota \rangle \simeq V_4 \trianglelefteq S_4$. So the map $U' \times_{\mathbb{Q}} \operatorname{Spec} \mathbb{Q}(i) \to M_g$ is Galois with Galois group $S_4/V_4 \simeq S_3$.

Finally, for (c) we just carry out parts (a) and (b) explicitly. For (a), we ask for the values of $b = b_1$ such that there is a new transformation (3.4.3) permuting S, giving

$$
\begin{aligned}
& x^{24} - 20x^{20} - 475x^{16} + 475x^8 + 20x^4 - 1 \\
(3.4.4) \quad & = (x^2+1)(x^4+1)(x^2-2x-1)(x^2+2x-1) \\
& \quad \cdot (x^4 - 2x^3 + 2x^2 + 2x + 1)(x^4 + 2x^3 + 2x^2 - 2x + 1)(x^4 + 6x^2 + 1).
\end{aligned}
$$

For (b), we apply the representatives of the 5 nontrivial classes in $S_3 \simeq S_4/V_4$ to $f(x)$ and compute the corresponding values of the parameter a. $\qquad\square$

4. Genus four

In this section, we mostly study a special case: the case $g = 4$ of the family in Setup 3.2.5 specialized to $a = 1/2$. (In section 4.4, we consider some consequences for other fibers in the family in genus $g = 4$.) We abbreviate

$$
C := \mathsf{C}_{1/2}^{(4)} : y^2 = x(x^4 - 1)(x^4 + x^2 + 1)
$$

and let $A := \mathsf{A}_{1/2}^{(4)} = \operatorname{Jac} \mathsf{C}_{1/2}^{(4)}$ be its Jacobian. The discriminant of C is $2^{28}3^6$, so C is nice of genus 4 and A has good reduction away from 2 and 3.

4.1. **Endomorphism algebra.** Running the algorithm of Costa–Mascot–Sijsling–Voight [CMSV18], we obtain the following theorem.

Theorem 4.1.1. *The Jacobian A has geometric endomorphism algebra*

$$(\text{End } A^{\text{al}})_{\mathbb{Q}} \simeq \left(\frac{-1,-1}{\mathbb{Q}}\right);$$

in particular, A is geometrically simple.

Proof. For the reader's convenience, we provide just a bit of detail on the mechanism which underlies the algorithm [CMSV18, §7], see also [Lom19].

We begin by computing the (geometric) endomorphism algebra of the reduction $A_{\mathbb{F}_p}$ of A modulo a good prime p [CMSV18, Lemma 7.2.4], as follows. Let

$$c_p(T) := \det(1 - \text{Frob}_p T \mid H^1_{\text{ét}}(A_{\mathbb{F}_p^{\text{al}}}, \mathbb{Q}_\ell)) \in 1 + T\mathbb{Z}[T]$$

be the characteristic polynomial of the p-power Frobenius endomorphism Frob_p acting on étale cohomology (independent of $\ell \neq p$), computed as the numerator of the zeta function of $A_{\mathbb{F}_p}$ by counting points. Let $c_p^{\otimes 2}(T) = \text{Res}_z(c(z), z^{2g}c(T/z))$ (the characteristic polynomial of Frobenius acting on $(H^1)^{\otimes 2}$). We factor

$$(4.1.2) \qquad c_p^{\otimes 2}(T) = h(pT) \prod_{i=1}^n \Phi_{k_i}(pT)$$

where Φ_{k_i} is the k_i-th cyclotomic polynomial and no root of $h(pT)$ is a root of unity. Then

$$(4.1.3) \qquad \dim_{\mathbb{Q}} \text{End}(A_{\mathbb{F}_p^r})_{\mathbb{Q}} = \sum_{k_i \mid r} \deg \Phi_{k_i},$$

and the endomorphism field of $A_{\mathbb{F}_p}$ is $\mathbb{F}_{p^k}$ where k is the least common multiple of the k_i.

Let $p = 41$. Computing using Magma [BCP97], we find that $c_p(T) = g_p(T)^2$ where

$$(4.1.4) \qquad g_p(T) = 1 - 2T - 30T^2 - 2pT^3 + p^2T^4$$

and

$$(4.1.5) \qquad c_p^{\otimes 2}(T) = h(pT)(1 - pT)^{16}$$

with $h(T)$ as in (4.1.2). Thus all endomorphisms of $A_{\mathbb{F}_p}^{\text{al}}$ are already defined over $\mathbb{F}_p$ and $\dim_{\mathbb{Q}} \text{End}(A_{\mathbb{F}_p})_{\mathbb{Q}} = 16$. (By Honda–Tate theory [CMSV18, Remark 7.2.13], since $c_p(T)$ is the square of an irreducible polynomial $g_p(T)$ with middle coefficient not divisible by p, we conclude that $A_{\mathbb{F}_p}$ is isogenous (over $\mathbb{F}_p$) to the square of a geometrically simple, ordinary abelian surface over $\mathbb{F}_p$ with endomorphism algebra $\mathbb{Q}(\pi) = \mathbb{Q}[T]/(g_p(T))$.)

We now match certain lower and upper bounds. Since A^{al} has quaternionic multiplication (3.2.9), we have

$$\eta(A^{\text{al}}) \geq 2 \cdot 1^2 \cdot 4 = 8$$

[CMSV18, (7.3.16)] and

$$\eta(A_{\mathbb{F}_p^{\text{al}}}) = 2^2 \cdot 4 = 16 = 2\eta(A^{\text{al}})$$

[CMSV18, (7.3.18)]. Therefore [CMSV18, Corollary 7.3.19(b)–(c)] (from $(e_1 n_1, n_1 \dim A_1) = (2, 4)$) we conclude that either A^{al} is simple (with geometric endomorphism algebra a quaternion algebra over $\mathbb{Q}$) or isogenous to the square of a simple abelian surface whose endomorphism algebra is commutative (a field). (This refines the conclusion that A^{al} is simple or isogenous to the product of two abelian surfaces, which follows from the injectivity of endomorphisms under reduction modulo p.)

To rule out the latter, we first compute the center $B = \mathrm{End}(A^{\mathrm{al}})_{\mathbb{Q}}$ [CMSV18, §7.4]. We repeat the calculation above for $p = 73$, and we get

$$c_{73}(T) = g_{73}(T)^2 = (1 + 8T - 2T^2 - 8pT^3 + p^2T^4)^2.$$

We compute that the only subfield common to $\mathbb{Q}[T]/(g_{41}(T))$ and $\mathbb{Q}[T]/(g_{73}(T))$ is $\mathbb{Q}$, so $Z(B) = \mathbb{Q}$ [CMSV18, Corollary 7.4.4]. (It is no coincidence that looking at two primes is enough [CLV21].) So we cannot have A^{al} isogenous to the square of an abelian surface: for then its endomorphism algebra would be $B = \mathrm{M}_2(L)$ where $L = Z(B) = \mathbb{Q}$, but we have an embedding $(-1, -1 \,|\, \mathbb{Q}) \hookrightarrow B$, a contradiction. $\square$

Corollary 4.1.6. *The Jacobian A has endomorphism field $\mathbb{Q}(\mathrm{End}\, A) = \mathbb{Q}(i)$ and connected monodromy field $\mathbb{Q}(\varepsilon_A) = \mathbb{Q}(\zeta_8)$.*

Proof. By Theorem 4.1.1, the endomorphism algebra of A is generated by the automorphisms of C, which by inspection are minimally defined over $\mathbb{Q}(i)$, so $\mathbb{Q}(\mathrm{End}\, A) = \mathbb{Q}(i)$.

By (1.1.5) we have $\mathbb{Q}(i) \subseteq \mathbb{Q}(\varepsilon_A)$ and by Theorem 2.3.4 we have $[\mathbb{Q}(\varepsilon_A) : \mathbb{Q}(i)] \leq 2$. We first show that equality holds. Let $p = 13$ and let $\mathfrak{p}$ be one of the two primes of $\mathbb{Q}(i)$ lying over p. We compute

$$\det(1 - \rho_{A,\ell}(\mathrm{Frob}_{\mathfrak{p}})T) = \det(1 - \rho_{A,\ell}(\mathrm{Frob}_p)T) = g_p(T)^2$$

where

$$g_p(T) = 1 - 2T + 2pT^3 - p^2T^4$$

so recalling (2.5.4) we get $\omega(\mathrm{Frob}_{\mathfrak{p}}) = -1$ and then by Lemma 2.5.6 we have $\rho_{A,\ell}(\mathrm{Frob}_{\mathfrak{p}}) \notin G^0_{A,\ell}$ (for any ℓ), so $[\mathbb{Q}(\varepsilon_A) : \mathbb{Q}(i)] = 2$.

To determine this quadratic extension, we use Lemma 2.5.7: ω can only be ramified at places of bad reduction of A, hence it is ramified at most at the places of $\mathbb{Q}(i)$ lying over 2 and 3. This means that $\mathbb{Q}(\varepsilon_A) = \mathbb{Q}(i)(\sqrt{\beta})$ with $\beta \in \langle i, 1+i, 3 \rangle$, the multiplicative subgroup of $\mathbb{Q}(i)^{\times}/\mathbb{Q}(i)^{\times 2}$ generated by the units and the (possibly) ramified primes. The condition that $\mathbb{Q}(\varepsilon_A)$ is Galois over $\mathbb{Q}$ implies that $\beta \in \langle i, 3 \rangle$, so the field $\mathbb{Q}(\varepsilon_A)$ is one of $L_1 = \mathbb{Q}(\sqrt{i}) = \mathbb{Q}(\zeta_8)$, $L_2 = \mathbb{Q}(i, \sqrt{3})$, $L_3 = \mathbb{Q}(i, \sqrt{3i})$. Finally, since 13 splits completely in both L_2 and L_3, for any prime $\mathfrak{P}$ above 13 in L_2 or L_3, we would again have $\omega(\mathrm{Frob}_{\mathfrak{P}}) = \omega(\mathrm{Frob}_p) = -1$, contradicting that $G_{A_{L_i},\ell}$ is connected; so we must have $\mathbb{Q}(\varepsilon_A) = L_1 = \mathbb{Q}(\zeta_8)$. $\square$

Corollary 4.1.6 shows that Theorem 2.3.4 is sharp, in general.

Remark 4.1.7. The proof of Theorem 4.1.1 ultimately relies on explicit calculations at the primes 41 and 73. Accordingly, for all $a \in \mathbb{Z}$ such that $a \equiv 1 \pmod{41 \cdot 73}$, the same conclusion holds for the Jacobian A_a, already giving not just one example but infinitely

many. (For Corollary 4.1.6 we also use $p = 13$, but we also need to restrict the primes of bad reduction.)

These primes might look comparatively large, but in fact they are among the smallest choices that are compatible with what we now know about A: we need primes that are totally split in $\mathbb{Q}(\varepsilon_A) = \mathbb{Q}(\zeta_8)$ (congruent to 1 modulo 8, the smallest are $17, 41, 73$). For Corollary 4.1.6, we need primes that split in $\mathbb{Q}(\mathrm{End}\, A) = \mathbb{Q}(i)$ but not $\mathbb{Q}(\varepsilon_A)$ (so congruent to 5 modulo 8, the smallest are $5, 13$).

Before moving on, we pause briefly to obtain a similar result for $g = 6$.

Proposition 4.1.8. *For $g = 6$ and $a = (a_1, a_2) = (1/2, 0)$, we have $(\mathrm{End}\, \mathsf{A}_a^{\mathrm{al}})_{\mathbb{Q}} \simeq (-1, -1 \,|\, \mathbb{Q})$ and $\mathbb{Q}(\mathrm{End}\, \mathsf{A}_a) = \mathbb{Q}(i)$.*

Proof. We perform entirely analogous computations: in this case for $p = 17$ we get

$$g_{17}(T) = 1 - 2T - 13T^2 + 44T^3 - 13pT^4 - 2p^2T^5 + p^3T^6$$

and for $p = 41$ we get

$$g_{41}(T) = 1 - 14T + 91T^2 - 540T^3 + 91pT^4 - 14p^2T^5 + p^3T^6$$

both with factorizations like (4.1.5), and no common subfield. $\square$

4.2. **Homology.** In this section, we describe the integral homology of C over $\mathbb{C}$. Let

$$S = \{0, 1, \zeta_6, i, \zeta_3, -1, -\zeta_6, -i, -\zeta_3, \infty\} \subseteq \mathbb{P}^1(\mathbb{C}).$$

The coordinate function $x\colon C_{\mathbb{C}} \to \mathbb{P}^1(\mathbb{C})$ realises $C_{\mathbb{C}}$ as a 2-to-1 cover of $\mathbb{P}^1(\mathbb{C})$, branched over the points in S. We consider paths $\delta_0, \ldots, \delta_4$ in the complex plane $\mathbb{C}$, thought of as the x-plane, as described in Figure 1.

FIGURE 1.   Image of the homology basis on the x-plane

FIGURE   2. Homology basis for X

The inverse image in $C_{\mathbb{C}}$ of each δ_i is homeomorphic to the union of two closed intervals with common endpoints, so it is a copy of $\mathbb{S}^1$. We parametrize each of these copies of $\mathbb{S}^1$ with functions $\gamma_i : [0,1] \to X$. To fix the orientation, we require that $y \circ \gamma_i(t)$ has positive imaginary part for $t \in [0,1]$ sufficiently close to 0; this uniquely determines the paths γ_i. Notice that $\iota \circ \gamma_i$ is the path $-\gamma_i$, that is, γ_i with the opposite orientation. Recall the automorphism α from (3.2.9). By standard results on the homology of real orientable surfaces, the eight paths

$$(4.2.1) \qquad \gamma_0, \alpha\gamma_0, \gamma_1, \alpha\gamma_1, \gamma_2, \alpha\gamma_2, \gamma_3, \alpha\gamma_3$$

form a basis of the integral homology $H_1(C_{\mathbb{C}}, \mathbb{Z})$, see also Figure 2. The action of α in terms of this basis is immediate to write down. Together with a straightforward calculation for β, this yields the matrices representing the action of α, β on homology:

(4.2.2)

$$\alpha_* = \begin{pmatrix} 0 & -1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & -1 & & & & \\ & & 1 & 0 & & & & \\ & & & & 0 & -1 & & \\ & & & & 1 & 0 & & \\ & & & & & & 0 & -1 \\ & & & & & & 1 & 0 \end{pmatrix}, \quad \beta_* = \begin{pmatrix} 1 & 0 & -1 & -1 & & & & \\ 0 & -1 & -1 & 1 & & & & \\ 1 & 1 & 0 & -1 & & & & \\ 1 & -1 & -1 & 0 & & & & \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 & -1 & 0 & 0 \\ 1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, the Gram matrix of the intersection form is given by

$$(4.2.3) \qquad \begin{pmatrix} 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix};$$

apart from the signs, the structure of this matrix can easily been gleaned from Figure 2.

4.3. **Complex lattice.** We now discuss the period lattice Λ attached to C. Let $c_1, \ldots, c_8$ be the (ordered) basis of the integral homology of C in (4.2.1). The period matrix of the abelian variety $A = \mathrm{Jac}(C)$ (over $\mathbb{C}$) with respect to this homology basis and to the basis $\{x^{\ell-1}\,\mathrm{d}x/y\}_{\ell=1,\ldots,4}$ for the regular differentials is by definition the 4×8 matrix

$$\Pi = (\Pi_{\ell j})_{\substack{\ell=1,\ldots,4 \\ j=1,\ldots,8}} = \int_{c_j} \frac{x^{\ell-1}\,\mathrm{d}x}{y}.$$

The columns of Π span a lattice $\Lambda \simeq \mathbb{Z}^8$ in $\mathbb{C}^4$, and we have an isomorphism $\mathbb{C}^4/\Lambda \simeq A(\mathbb{C})$ of complex abelian varieties. Clearly the lattice Λ carries an action of the Lipschitz order $\mathcal{O}$: in fact, we will show that equality holds (and more) below.

This complex description comes with two representations of the endomorphism ring. First is the *analytic* (or *tangent*) *representation* $M(\varphi)$ obtained by letting $\varphi \in \mathrm{End}\,A^{\mathrm{al}}$ act on the above basis for the space of regular differentials. With respect to the natural embedding $\mathrm{Aut}\,C^{\mathrm{al}} \hookrightarrow \mathrm{End}\,A^{\mathrm{al}}$,

$$(4.3.1) \qquad M(\alpha) := i \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \text{ and } M(\beta) := i \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}.$$

Along the way, since

$$(4.3.2) \qquad \alpha^* \left(\frac{x^{\ell-1}\,\mathrm{d}x}{y} \right) = (-1)^{\ell+1} \frac{x^{\ell-1}\,\mathrm{d}x}{y}$$

we also conclude that α^* in (4.2.2) also provides the complex structure on Λ, i.e., the action of α^* on $\Lambda \subset \mathbb{C}^4$ corresponds to multiplication by i.

The *rational representation* of an endomorphism $\varphi \in \mathrm{End}(J)$ is the unique matrix $R(\varphi) \in \mathrm{Mat}_{8\times8}(\mathbb{Z})$ such that

$$M(\varphi)\Pi = \Pi R(\varphi),$$

Given the definition of the period matrix, it is easy to see that for an endomorphism of A induced by an automorphism φ of X the matrix $R(\varphi)$ coincides with the matrix of φ_* acting on $H_1(X, \mathbb{Z})$ (with respect to the homology basis used to construct the period

matrix). In particular, since we are using the basis of integral homology given in (4.2.1), the matrices $R(\alpha)$ and $R(\beta)$ coincide with those given in (4.2.2).

Proposition 4.3.3. *We have* $\operatorname{End} A^{\mathrm{al}} = \mathcal{O}$ *and the* $\mathcal{O}$*-module* Λ *is not free.*

Proof. For the first statement (generalized in Lemma 5.1.1), recall that the unique maximal order containing the Lipschitz order is the Hurwitz order, generated by the element $\omega = (-1 + i + j + ij)/2$ over $\mathcal{O}$. But from the rational representation (4.2.2), we see that $R(\omega) \notin \mathrm{M}_8(\mathbb{Z})$, a contradiction.

For the second statement, in fact we will show that $\Lambda_2 := \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_2 \simeq \mathbb{Z}_2^8$ is not free over $\mathcal{O}_2 \simeq \mathbb{Z}_2^4$, so Λ is not even locally free! The Jacobson radical (see Voight [Voi21, §11.1, §§20.4–20.5, §24.3]) of $\mathcal{O}_2$ is $J_2 := \mathrm{rad}(\mathcal{O}_2) = (1 + i, 1 + j, 1 + k)$. By Nakayama's lemma, it suffices to show that $\Lambda_2/J_2\Lambda_2$ is not free over $\mathcal{O}_2/J_2 \simeq \mathbb{F}_2$. Since $J_2/2\mathcal{O}_2$ is spanned by $1 + i, 1 + j, 1 + k$ as an $\mathbb{F}_2$-vector space, $J_2\Lambda_2/2\Lambda_2$ is spanned over $\mathbb{F}_2$ by the columns of the matrices $1 + R(i), 1 + R(j), 1 + R(k)$. Given the explicit form of the matrices $R(i)$ and $R(j)$ given in (4.2.2), we compute that $\dim_{\mathbb{F}_2}(J_2\Lambda_2/2\Lambda_2) = 5$ so

$$\dim_{\mathbb{F}_2}(\Lambda_2/J_2\Lambda_2) = \dim_{\mathbb{F}_2}(\Lambda_2/2\Lambda_2) - \dim_{\mathbb{F}_2}(J_2\Lambda_2/2\Lambda_2) = 8 - 5 = 3.$$

So if Λ_2 was free over $\mathcal{O}_2$, it would be free of rank 3, so $\Lambda_2 \simeq \mathcal{O}_2^3 \simeq \mathbb{Z}_2^{12}$, a contradiction. $\qquad\square$

Remark 4.3.4. The lack of freeness of Λ may seem undesirable, but we cannot have it all: see Birkenhake–Lange [BL04, Exercise 9.10(2)] (and more generally Shimura [Shi63]) for a criterion which determines when an abelian variety of Type III is geometrically simple or not. Indeed, if Λ is free of rank 2 over $\mathcal{O}$ and equipped with a natural principal polarization, then necessarily $\mathbb{C}^4/\Lambda$ is isomorphic to the square of an abelian surface [BL04, Example 9.5.6].

4.4. Conclusions about the family.
Although we will pursue this more generally in the next sections, we pause to conclude with a few remarks about our family in genus $g = 4$. We have

$$(4.4.1) \qquad \mathsf{C}_a^{(4)} \colon y^2 = f(x) = x(x^4 - 1)(x^4 + 2ax^2 + 1).$$

The discriminant of $f(x)$ is $\Delta = -2^{12}(a - 1)^6(a + 1)^6$, so we obtain a nice family of genus 4 curves over $\mathbb{A}^1 \smallsetminus \{\pm 1\}$.

Some of the results we proved for a specialization can be extended as follows.

- Since endomorphism rings inject under specialization, from Theorem 4.1.1 we conclude that the generic geometric endomorphism algebra of $\mathsf{A}^{(4)}$ is $(-1, -1 \mid \mathbb{Q})$ and accordingly the generic endomorphism field is $\mathbb{Q}(i)$.
- Since the rational representations (4.2.2) and intersection form (4.2.3) are a discrete invariant, they are constant over the continuous parameter $a \in \mathbb{C} \smallsetminus \{\pm 1\}$. Since we kept a fixed basis of holomorphic differentials, the analytic representations (4.3.1) are also constant.
- The conclusion of Proposition 4.3.3 then holds for all $a \in \mathbb{C}$ such that $(\operatorname{End} \mathsf{A}_a^{\mathrm{al}})_{\mathbb{Q}} = (-1, -1 \mid \mathbb{Q})$, in particular it holds generically.

Another interesting feature (not used in the sequel) of the family in genus $g = 4$ is the following.

Lemma 4.4.2. *Let K be a number field and let $a \in K \smallsetminus \{\pm 1\}$. Then the abelian variety A_a over K has everywhere potentially good reduction away from the primes of $\mathbb{Z}_K$ above 2.*

Proof. We follow the proof of Oort [Oor88, Proof of Lemma (3.9)]. Let $\mathfrak{p}$ be a nonzero prime of $\mathbb{Z}_K$ and let $K_{\mathfrak{p}}$ be the completion of K at $\mathfrak{p}$. Then there is an extension $L_{\mathfrak{P}} \supseteq K_{\mathfrak{p}}$ such that $(\mathsf{A}_a)_{L_{\mathfrak{P}}}$ has semistable reduction. Let $\mu \leq 4$ be the dimension of the toric part. If $\mu = 0$, then A_a obtains good reduction over $L_{\mathfrak{P}}$ as desired; so suppose $\mu > 0$. Then over $L_{\mathfrak{P}}(i)$ we have an embedding $B = (-1, -1 \mid \mathbb{Q}) \hookrightarrow \mathrm{M}_{\mu}(\mathbb{Q})$. Since B is a division algebra, we must have $\mu = 4$, which is to say A_a has purely toric reduction over $L_{\mathfrak{P}}$. We show that this is impossible when $\mathfrak{P}$ does not lie above 2, arguing directly with the equation for C_a: modulo $\mathfrak{P}$, the subset $\{\pm b, \pm 1/b\}$ reduces to a subset which can only intersect with the set $\{0, \infty, \pm 1, \pm i\}$ in at most two points, so the reduction of this set of roots has at least 4 distinct elements, so the normalization of the reduction of the curve C_a has normalization of genus $g \geq 1$ and hence the Jacobian cannot have purely toric reduction. $\square$

Remark 4.4.3. Neither the statement nor the proof of Lemma 4.4.2 generalizes to $g \geq 6$: see Remark 6.6.4.

For $g = 4$, it would be interesting to know if Lemma 4.4.2 extends to $p = 2$: do the abelian varieties A_a also have potential good reduction at 2?

Remark 4.4.4. A more conceptual proof of Lemma 4.4.2 (still only for primes $p \neq 2$) can be obtained by looking at the action of the inertia group on the 2-adic Tate module: by Grothendieck [Gro72, Theorem 2.4], if the toric rank is 4, then there is a free rank-4 submodule that is stable under the endomorphism ring and totally isotropic for the Weil pairing on which the inertia group acts trivially. It is possible to verify explicitly that no such submodule exists.

5. Endomorphisms and classes

In this section, we establish results on endomorphisms and algebraic classes on the Jacobians in our family, under certain hypotheses. In the next section, we will combine these with an inductive argument to prove our main result, Theorem 1.2.4. Throughout, we work with with Setup 3.2.5: let K be a number field and let $a = (a_1, \ldots, a_d) \in U(K)$.

5.1. Endomorphism ring and endomorphism field. We get started by showing that properties (i) and (ii) in Theorem 1.2.4 on the endomorphism ring is implied by the (formally weaker) statement concerning the endomorphism algebra.

Lemma 5.1.1. *Suppose that the Jacobian A_a of C_a has geometric endomorphism algebra $(-1, -1 \mid \mathbb{Q})$. Then its geometric endomorphism ring is $\mathrm{End}\,\mathsf{A}_a^{\mathrm{al}} = \mathcal{O}$ and its endomorphism field is $K(i)$.*

More precisely, let α, β be the automorphisms of C_a given in (3.2.9), and let α^*, β^* be the corresponding automorphisms of A_a. Then the inclusion $\mathcal{O} \hookrightarrow \mathrm{End}\,\mathsf{A}_a^{\mathrm{al}}$ induced by $i, j \mapsto \alpha^*, \beta^*$ is an isomorphism.

Proof. We start with $\mathcal{O} \hookrightarrow \operatorname{End} \mathsf{A}_a^{\mathrm{al}} \subset (\operatorname{End} \mathsf{A}_a^{\mathrm{al}})_{\mathbb{Q}} = (-1, -1 \,|\, \mathbb{Q})$, and for simplicity we identify $i = \alpha^*$, $j = \beta^*$, and $k = ij = \alpha^*\beta^*$ with their images. The endomorphism field is $K(i)$ since this is the minimal field of definition of i, j.

The Lipschitz order $\mathcal{O}$ is contained in a unique maximal order of index 2, the Hurwitz order $\mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}\omega$, where $\omega = (-1+i+j+k)/2$. So to prove the lemma, we need only show that $\omega \notin \operatorname{End} \mathsf{A}_a^{\mathrm{al}}$; that is to say, that the element $2\omega = -1+i+j+k$ is not divisible by 2 in $\operatorname{End} \mathsf{A}_a^{\mathrm{al}}$, which is equivalent to it acting nontrivially on the 2-torsion $A^{\mathrm{al}}[2]$. To this end, we recall section 3.3, in particular the calculation of the action of i, j on $A^{\mathrm{al}}[2]$ (3.3.7)–(3.3.8). We then verify directly that 2ω acts nontrivially, as desired. $\square$

5.2. Connected monodromy field: lower bounds. In this section we prove the following lower bound on the connected monodromy field.

Theorem 5.2.1. *Suppose $i \in K$ and $a_d \neq 0$. Then $H^g(\mathsf{A}_a, \mathbb{Q}_\ell(g/2))$ contains algebraic classes whose field of definition is $K(\zeta_8)$. In particular, the connected monodromy field of A_a contains $K(\zeta_8)$.*

Example 5.2.2. Theorem 5.2.1 justifies the computation of the connected monodromy field in Zywina [Zyw20, Example 1.8]. This example is in our family in genus $g = 10$, with parameter $a = (7, 1, 7, 1/2)$. We recall (1.1.5) that $\mathbb{Q}(\operatorname{End} A) = \mathbb{Q}(i) \subseteq \mathbb{Q}(\varepsilon_A)$. By Theorem 2.3.4 over $K = \mathbb{Q}(i)$, the degree $[K(\varepsilon_A) : K(\operatorname{End} A)]$ is at most 2. Now Theorem 5.2.1 implies that $K(\varepsilon_A) = \mathbb{Q}(\zeta_8)$.

The proof of Theorem 5.2.1 will occupy this section: we will construct some nontrivial cohomology classes in $H^g_{\text{ét}}(\mathsf{A}_a, \mathbb{Q}_\ell(g/2))$ (with ℓ an auxiliary prime) and show that they are minimally defined over $K(\zeta_8)$. This implies Theorem 5.2.1, with the final statement following from Proposition 2.4.3. We follow closely Schoen [Sch88, Proof of Theorem 2.0].

We begin by finding a (singular) model of C_a on which the action of an automorphism of order 4 takes a particularly simple form. Up to birational equivalence, we may take several models for C_a of the form

$$(5.2.3) \qquad z^4 = f_\ell(t) := t(t^2-1)^2 \left(t^{g-2} + 1 + \sum_{i=1}^{d} a_i(t^{g-2-i} + t^i) \right)^2 (t - \gamma_\ell)^{-2g-4},$$

where $t = x^2$, $z = y \cdot (t - \gamma_\ell)^{-g/2-1}$ and the elements $\gamma_\ell \in K$, for $\ell = 1, \ldots, g+1$, will be chosen below. Note that the field $K(t, z)$ coincides with $K(x, y)$, so (5.2.3) is indeed a model of C_a. The automorphism α of C_a given in (3.2.9) corresponds to

$$(5.2.4) \qquad\qquad\qquad\qquad \sigma : (t, z) \mapsto (t, iz).$$

The quotient $P = \mathsf{C}_a/\langle\sigma\rangle$ is isomorphic to $\mathbb{P}^1$, with the quotient map $\psi : \mathsf{C}_a \to \mathbb{P}^1$ given by $(x, y) \mapsto x^2$, which corresponds to $(t, z) \mapsto t$. In particular, ψ has $g + 2$ branch points, the roots of $f_\ell(t)$ together with ∞. We choose $\gamma_1, \ldots, \gamma_{g+1} \in K$ to be distinct and different from all of these branch points. The topological Euler characteristic of $\mathbb{P}^1 \setminus \{f_\ell = 0\}$ is therefore $2 - (g+2) = -g$, and so the invariant denoted h by Schoen [Sch88] is equal to g [Sch88, Lemma 1.5].

Write $N := \{(\sigma_1, \ldots, \sigma_g) \in (\mathbb{Z}/4\mathbb{Z})^g : \sum_i \sigma_i = 0\}$ and let G be the subgroup of $\mathrm{Aut}(\mathsf{C}_a^g)$ generated by N and the group S_g acting by permuting factors. In our situation, [Sch88, Diagram (2.1)] reads

(5.2.5)
$$
\begin{array}{ccc}
\mathsf{C}_a^g & \longrightarrow & \mathrm{Sym}^g\,\mathsf{C}_a \\
\downarrow & & \downarrow \\
N \backslash \mathsf{C}_a^g & \longrightarrow & W_0 \\
\downarrow & & \downarrow{\scriptstyle \pi_0} \\
P^g & \xrightarrow{\ \gamma_P\ } & \mathrm{Sym}^g\,P,
\end{array}
$$

where $W_0 := G \backslash \mathsf{C}_a^g$ and all arrows except $\mathrm{Sym}^g\,\mathsf{C}_a \to W_0$ are quotient maps for the obvious (finite) group actions. It will be useful to identify $\mathrm{Sym}^g\,P \simeq \mathrm{Sym}^g\,\mathbb{P}^1$ with the projectivization of the space of homogeneous polynomials of degree g in two variables X, Y. In these coordinates, the horizontal map γ_P becomes

(5.2.6)
$$
\gamma_P : \quad
\begin{array}{ccc}
(\mathbb{P}^1)^g & \to & \mathrm{Sym}^g\,\mathbb{P}^1 \\
([x_1 : y_1], \ldots, [x_g : y_g]) & \mapsto & [(y_1 X - x_1 Y) \cdots (y_g X - x_g Y)].
\end{array}
$$

We may consider $f_\ell(t)$ as a function on $\mathbb{P}^1$ and write its divisor as

$$
(f_\ell) = 2(1) + 2(-1) + (0) + 3(\infty) + 2\left(\sum_{i=1}^{d}(\beta_i) + \sum_{i=1}^{d}(1/\beta_i)\right) + 4D_\ell,
$$

where $D_\ell := -((g/2) + 1)(\gamma_\ell)$. We order the points $b_1, \ldots, b_{g+2}$ that appear in the support of (f_ℓ) with positive coefficient as $1, -1, 0, \infty$, followed by the $g - 2$ roots of $t^{g-2} + 1 + \sum_{i=1}^{d} a_i(t^{g-2-i} + t^i)$ (in any order). We will denote these roots by $\beta_1, 1/\beta_1, \ldots, \beta_d, 1/\beta_d$, so that

(5.2.7)
$$
t(t^2 - 1)^2\left(t^{g-2} + 1 + \sum_{i=1}^{d} a_i(t^{g-2-i} + t^i)\right)^2 = t(t^2 - 1)^2 \prod_{j=1}^{d}(t - \beta_j)^2(t - 1/\beta_j)^2.
$$

Notice that it is not necessarily the case that each point b_i is rational over K, but the sum $\sum_{j=5}^{g+2}(b_j)$ is certainly stable under Gal_K.

To make it easier to make the comparison with Schoen's article, we point out that, in the notation of the equation $(f_\ell) = \sum_{1 \le j \le 2r} \alpha_j b_j + mD_\ell$ [Sch88, bottom of page 12], we have $m = 4$, $r = g/2 + 1$, $\alpha_j = 2$ for all $j \neq 3, 4$, $\alpha_3 = 1$, and $\alpha_4 = 3$.

The branch locus of π_0 is the union of the linear subspaces $\gamma_P(b_j \times P^{g-1})$ of $\mathrm{Sym}^g\,P \simeq \mathbb{P}^g$, for $j = 1, \ldots, g+2$ [Sch88, Lemma 2.2]. This union is defined over K. Looking at (5.2.6), one sees immediately that $\gamma_P(b_j \times (\mathbb{P}^1)^{g-1})$ is the linear subspace $S_j \subseteq \mathbb{P}^g$ of homogeneous polynomials that vanish at b_j. Following [Sch88, Page 13], let $V_i = \bigcap_{j \neq 2i-1, 2i} S_j$ for $i = 1, \ldots, g/2 + 1$ (notice that, in our setting, (P_∞) is the whole $\mathrm{Sym}^g\,P$ since P is of genus 0). Each V_i is a single point: it is the projective point corresponding to the one-dimensional linear subspace of degree-g homogeneous polynomials that vanish at the g points $\{b_j : j \neq$

$2i - i, 2i$}. For $i = 1, \ldots, g/2+1$ we can write down an obvious explicit generator $q_i(X,Y)$ of the linear subspace corresponding to V_i: homogenizing, we see that we may take

(5.2.8)
$$q_1(X,Y) = XY \prod_{j=1}^{d}(X - \beta_j Y)(X - 1/\beta_j Y),$$

$$q_2(X,Y) = (X^2 - Y^2) \prod_{j=1}^{d}(X - \beta_j Y)(X - 1/\beta_j Y),$$

and

(5.2.9)
$$q_i(X,Y) = XY(X^2 - Y^2) \prod_{j \neq i}(X - \beta_j Y)(X - 1/\beta_j Y).$$

Notice that $q_1(X,Y)$ is symmetric under the exchange $X \leftrightarrow Y$, while $q_i(X,Y)$ is anti-symmetric for $i \geq 2$. Once again following [Sch88, Page 13], we let P_0 denote the projective span of the points $V_1, \ldots, V_{g/2+1}$; then P_0 has the expected dimension $g/2$ [Sch88, Lemma 2.3(v)]. We now describe the linear equations (with rational coefficients) that vanish on the subspace P_0. We let $Z_0, \ldots, Z_g$ denote linear coordinates on the space of homogeneous polynomials of degree g in X, Y, where Z_i is the coefficient of $X^i Y^{g-i}$. By a slight abuse of language, we also denote by $Z_0, \ldots, Z_g$ the projective coordinates that these functions induce on $\mathrm{Sym}^g(P) \simeq \mathbb{P}^g$.

Recall that $q_2, \ldots, q_{g/2+1}$ are anti-symmetric polynomials, so they satisfy $(Z_j + Z_{g-j})(q_i) = 0$ (in particular, $Z_{g/2}(q_i) = 0$) for $j = 0, \ldots, g$ and $i = 2, \ldots, g/2+1$. The quantity $Z_{g/2}(q_1)$ is nonzero: indeed, this is the coefficient of $X^{g/2-1}Y^{g/2-1}$ in

$$\prod_{j=1}^{d}(X - \beta_j Y)(X - 1/\beta_j Y) = X^{g-2} + Y^{g-2} + \sum_{i=1}^{d} a_i(X^{g-2-i}Y^i + X^i Y^{g-2-i}),$$

so it is equal to $2a_d \neq 0$ by assumption. It follows that for $j = 0, \ldots, g$ there exists a unique $c_j \in K$ such that

(5.2.10)
$$(Z_j + Z_{g-j})(q_1) = c_j Z_{g/2}(q_1).$$

For $j = 0, \ldots, g/2 - 1$, we see that the $g/2$ homogeneous linear equations given by $Z_j + Z_{g-j} - c_j Z_{g/2} = 0$ are satisfied by $q_1, \ldots, q_{g/2+1}$, so that we get linear equations that vanish on P_0. Since these $g/2$ equations are clearly independent, they define a subspace of dimension $g/2 = \dim P_0$. We have proven the following lemma.

Lemma 5.2.11. *The following holds:*
$$P_0 = \{[Z_0 : \cdots : Z_g] \in \mathbb{P}^g : Z_j + Z_{g-j} = c_j Z_{g/2} \text{ for } j = 0, \ldots, g/2 - 1\}.$$

Remark 5.2.12. We remark that by symmetry we have $c_j = c_{g-j}$ for all $j = 0, \ldots, g$, and that $c_{g/2} = 2$. Also notice that we have $c_0 = c_g = 0$: indeed, $q_1(X,Y)$ has no term in X^g and no term in Y^g, because by construction it is divisible by XY.

The coefficients c_j also satisfy the following:

Lemma 5.2.13. *The equality*

$$\sum_{j=0}^{g} c_j u^j = \frac{2}{Z_{g/2}(q_1)} q_1(u, 1)$$

holds in the ring of polynomials $K[u]$.

Proof. By construction we have

$$c_j = \frac{(Z_j + Z_{g-j})(q_1)}{Z_{g/2}(q_1)},$$

so

(5.2.14) $$Z_{g/2}(q_1) \sum_{j=0}^{g} c_j u^j = \sum_{j=0}^{g} (Z_j + Z_{g-j})(q_1) u^j.$$

Recalling that $q_1(X, Y)$ is symmetric in X, Y, we have the tautological identity

$$\sum_{j=0}^{g} Z_j(q_1) u^j = q_1(u, 1)$$

and the claim follows. $\square$

Recall that t is a coordinate on P. We denote by $t_1, \ldots, t_g$ the corresponding coordinates on g copies of P. Let $\ell \in \{1, \ldots, g+1\}$ and let $f(t) := f_\ell(t)$, $\gamma := \gamma_\ell$, and

$$\hat{f} := \prod_{i=1}^{g} f(t_i),$$

which we may consider as an element of the function field $K(\mathrm{Sym}^g(P))$.

We now need a slightly involved, but completely algebraic computation; rather than inserting it here, we relegate it to its own section: in section 5.4 we show explicitly that

(5.2.15) $$-\hat{f}|_{P_0} = \hat{u}^4$$

for a suitable $\hat{u}$ in $K(P_0)$. It follows that the minimal field F containing K and such that $\hat{f}$ is a fourth power in $F(P_0)$ is $F = K(\sqrt[4]{-1}) = K(\zeta_8)$.

Then the key result of Schoen [Sch88, Theorem 2.0] gives nontrivial algebraic cohomology classes in $H^g(\mathrm{Sym}^g \mathsf{C}_a, \mathbb{Q}_\ell(g/2))$ which are defined [Sch88, Remark 2.13] over the smallest extension F' of K for which $\hat{f}$ is the fourth power of a function in $F'(P_0)$, that is, $F' = K(\zeta_8)$. Note that we are not yet claiming that F' is the *minimal* field of definition of these classes, even though this is true and will be checked below. More precisely, Schoen claims the desired conclusion only under the assumption that the branch points β_j be separately rational over K. However, the only place where this hypothesis is used in the argument is to ensure that the projective space P_0 be defined over K, which is true in our case even when the β_j are not rational over K, since we have explicitly found K-rational equations for P_0: the fact that P_0 is defined over K can be explained *a priori* because of the special structure of the ramification locus, in turn induced by the presence of a second automorphism of

order 4 of C_a. Finally, one observes [Sch88, Remark 1.4] that the cycles thus constructed on $\mathrm{Sym}^g C_a$ actually come (via the isomorphism Ξ^* [Sch88, Remark 1.4], composed with the isomorphism $\mathrm{Alb}(C_a) \simeq A_a$ given by the canonical principal polarization) from A_a—which, as it is well known, is birational to $\mathrm{Sym}^g C_a$.

It only remains to show that the cohomology classes in question are not defined over K when $\zeta_8 \notin K$. In order to do this we need to say something about the construction of these classes. Let

$$(5.2.16) \qquad \begin{aligned} \delta \colon \mathbb{Z}/4\mathbb{Z} &\to \mathrm{Aut}(C_a^g) \\ t &\mapsto (\sigma^t, \mathrm{id}, \dots, \mathrm{id}). \end{aligned}$$

Referring to (5.2.5), we may take as generators for the function field $K(\mathrm{Sym}^g(P))$ the elementary symmetric functions $e_1, \dots, e_g$ in the g variables $t_1, \dots, t_g$. So the function field of W_0 is $K(e_1, \dots, e_g, \hat{z})$ [Sch88, page 12], where

$$\hat{z}^4 = \prod_{i=1}^{g} f(t_i) = \hat{f}.$$

Notice that $\delta(t)$ descends to the automorphism of W_0 induced by $\hat{z} \mapsto i^t \hat{z}$, and that our computation of $\hat{f}|_{P_0}$ implies that $P_0 \times_{\pi_0} W_0$ consists of four geometrically irreducible components, each defined over $K(\zeta_8)$: the fibre of W_0 over the generic point of P_0 has equation $\hat{z}^4 = -\hat{u}^4$. If $\zeta_8 \notin K$, the extension $K(\zeta_8)/K$ has degree 2 (recall that by assumption K contains a primitive fourth root of unity): let τ be the nontrivial automorphism of $K(\zeta_8)/K$. The action of τ on the four irreducible components of $P_0 \times_{\pi_0} W_0$ exchanges them in pairs: this can be checked on the generic points, and the four components have equations $\hat{z} = \zeta_8^i \hat{u}$ for $i \in \{1, 3, 5, 7\}$. Letting Q_i denote the component corresponding to $\hat{z} = \zeta_8^i \hat{u}$, for $i \in \{1, 3, 5, 7\}$, we have $\tau(Q_i) = Q_{i+4 \bmod 8}$.

Let now χ be a primitive character of $\mathbb{Z}/4\mathbb{Z}$ and consider the cycle, with coefficients in $\mathbb{Q}(i)$, given by [Sch88, page 15]

$$z_\chi := \sum_{t \in \mathbb{Z}/4\mathbb{Z}} \chi(-t) \delta(t)_* Q_1.$$

Noticing that $\delta(t)_*(Q_1) = Q_{1-2t \bmod 8}$ (check this on generic points), we obtain that $\tau(z_\chi) = -z_\chi$, hence that the same holds for the cohomology class defined by z_χ. Then Schoen [Sch88, Lemma 2.6] shows that

$$(5.2.17) \qquad [z_\chi] \in H^g(W_0^{\mathrm{al}}, \mathbb{Q}_\ell(g/2)) = H^g((C_a^{\mathrm{al}})^g, \mathbb{Q}_\ell(g/2))^G \subset H^g(\mathrm{Sym}^g C_a^{\mathrm{al}}, \mathbb{Q}_\ell(g/2))$$

is nonzero. Since τ acts nontrivially on these nonzero classes (we have constructed two of them, corresponding to the two group isomorphisms $\chi : \mathbb{Z}/4\mathbb{Z} \to \{\pm 1, \pm i\}$), we have shown as desired that they are not defined over K. This completes the proof of Theorem 5.2.1.

For later use we also record the following lemma.

Lemma 5.2.18. *The subspace of $H^g(A_a(\mathbb{C}), \mathbb{C})$ consisting of algebraic classes is at least 2-dimensional.*

Proof. Combine results of Schoen [Sch88, Theorem 2.0, Lemma 1.2, and Remark 1.4] with the calculation above. □

5.3. **Algebraic classes.** In this section, we determine the connected monodromy field of A_a, under the assumption that its geometric endomorphism algebra is as small as possible and that its associated Galois representations are as large as possible.

Theorem 5.3.1. *Suppose that the following conditions hold:*

(i) $a_d \neq 0$,
(ii) $\mathrm{End}(\mathsf{A}_a^{\mathrm{al}})_{\mathbb{Q}} = (-1, -1 \,|\, \mathbb{Q})$ *(in particular, A_a is geometrically simple)*,
(iii) $\mathrm{Hg}(\mathsf{A}_a)_{\mathbb{C}} \simeq \mathrm{SO}_{g,\mathbb{C}}$, *and*
(iv) A_a *satisfies the Mumford–Tate conjecture.*

Then the following statements hold:

(a) *The connected monodromy field of A_a is $K(\zeta_8)$.*
(b) *The Tate conjecture and the Hodge conjecture are true for A_a and all its powers.*

We begin by a slightly more general discussion, so let A be any g-dimensional abelian variety over K. Suppose that $\mathrm{End}(A^{\mathrm{al}})_{\mathbb{Q}} = B = (-1, -1 \,|\, \mathbb{Q})$ and that $\mathrm{Hg}(A)_{\mathbb{C}} \simeq \mathrm{SO}_{g,\mathbb{C}}$. Let F be a quadratic imaginary subfield of B. The group $(B \otimes_{\mathbb{Q}} \mathbb{C})^{\times} \simeq \mathrm{GL}_2(\mathbb{C})$ acts on $H^1(A(\mathbb{C}), \mathbb{C})$. This action makes $H^1(A(\mathbb{C}), \mathbb{C})$ into a representation of $\mathrm{GL}_2(\mathbb{C})$ isomorphic to g copies of the standard 2-dimensional representation. Upon restriction to the subgroup $(F \otimes \mathbb{C})^{\times} \simeq (\mathbb{C}^{\times})^2 \subset \mathrm{GL}_2(\mathbb{C})$, the space $H^1(A(\mathbb{C}), \mathbb{C})$ decomposes as the direct sum of two subspaces Z_{σ} and $Z_{\overline{\sigma}}$ on which $x \in F^{\times}$ acts respectively via the conjugate embeddings $\sigma, \overline{\sigma} : F \hookrightarrow \mathbb{C}$. One can show that $\dim Z_{\sigma} = \dim Z_{\overline{\sigma}} = g$ (see Abdulali [Abd99, §4] or van Geemen–Verra [vGV03, Lemma 4.5]).

It is not hard to show that $\bigwedge^g(Z_{\sigma})$ and $\bigwedge^g(Z_{\overline{\sigma}})$, considered as subspaces of $\bigwedge^g(Z_{\sigma} \oplus Z_{\overline{\sigma}}) = \bigwedge^g H^1(A(\mathbb{C}), \mathbb{C}) = H^g(A(\mathbb{C}), \mathbb{C})$, are invariant under the action of the Hodge group, and therefore consist of Hodge classes. The subspace $\bigwedge^g(Z_{\sigma}) \oplus \bigwedge^g(Z_{\overline{\sigma}})$ of $H^g(A(\mathbb{C}), \mathbb{C})$ is defined over $\mathbb{Q}$ [vGV03, §4.3]. We denote by $W_F \subset H^g(A(\mathbb{C}), \mathbb{Q})$ the corresponding rational subspace and call it the space of Weil classes for the quadratic field F. Finally, we let W_B be the subspace of $H^g(A(\mathbb{C}), \mathbb{Q})$ spanned by $x \cdot W_F$ for $x \in B$. By [vGV03, Proposition 4.7], the subspace W_B is independent of the choice of the field F, consists of Hodge classes, and has dimension $g + 1$.

Let now $\mathcal{A}^{\bullet}$ be the the subalgebra of $H^{\bullet}(A(\mathbb{C}), \mathbb{C})$ given by Hodge classes and let $\mathcal{D}^{\bullet}$ be the subalgebra of $\mathcal{A}^{\bullet}$ generated by the classes of divisors. A straightforward computation with Hodge groups shows that $\mathcal{A}^2 = \mathcal{D}^2$ is 1-dimensional, with generator given by the class E of a polarization. Theorem 4.1 in [Abd99] shows that

$$(5.3.2) \qquad \mathcal{A}^{2i} = \langle \textstyle\bigwedge^i E \rangle \quad \text{for } i \neq g/2, \quad \mathcal{A}^g = \langle \textstyle\bigwedge^{g/2} E \rangle \oplus W_B.$$

Finally, the proof of [vGV03, Lemma 4.7] shows that $W_B \otimes \mathbb{C}$ is an irreducible submodule of $H^g(A(\mathbb{C}), \mathbb{C})$ for the action of $(B \otimes \mathbb{C})^{\times}$, and this clearly implies that W_B is an irreducible submodule for the action of $B^{\times}$.

Remark 5.3.3. The previous discussion shows that the Weil classes for a subfield F of B are essentially geometric incarnations of the determinant of a certain sub-F-representation of $H_1(\mathsf{A}_a(\mathbb{C}), \mathbb{C})$ of dimension g. The proof of Corollary 4.1.6 proceeds precisely by looking at a certain determinant (a coefficient of a reduced characteristic polynomial) and showing that it is not stabilised by the whole absolute Galois group of $\mathbb{Q}(i)$. Thus, the proof of Corollary 4.1.6 essentially amounts to explicitly verifying that certain Weil classes are not defined over $\mathbb{Q}(i)$.

Remark 5.3.4. Schoen [Sch88, Corollary 3.1] shows that the algebraic classes considered in section 5.2 are essentially the Weil classes for the subfield $\mathbb{Q}(i)$ of B, where i is identified with the automorphism of $\mathrm{Jac}(\mathsf{C}_a)$ induced by the automorphism σ of order 4 described in (5.2.4).

We can now prove Theorem 5.3.1.

Proof of Theorem 5.3.1. We specialize the above discussion to $A = \mathsf{A}_a$. The 2-dimensional linear subspace $U \subseteq H^g(\mathsf{A}_a(\mathbb{C}), \mathbb{C})$ of algebraic classes given by Lemma 5.2.18 cannot be contained in $\mathcal{D}^g$, which is 1-dimensional. Since algebraic classes are Hodge, this shows that $U \subset \mathcal{A}^g$ intersects W_B nontrivially. Moreover, the subspace of algebraic classes is stable under the action of $B = \mathrm{End}(\mathsf{A}_a^{\mathrm{al}})_{\mathbb{Q}}$, so the whole W_B (which is an irreducible B-module) consists of algebraic classes. Using (5.3.2) we have then proved that the Hodge conjecture holds for A_a, and since by assumption the Mumford–Tate conjecture is true for A_a we obtain that the Tate conjecture also holds for A_a [Moo17, Proposition 2.3.2].

We have proved that $\mathcal{A}^\bullet$ is generated by $\mathcal{D}^\bullet$ and by the translates of Weil classes under the action of endomorphisms. In particular, the minimal field of definition of the (ℓ-adic) cohomology classes of all the algebraic cycles on A_a is the minimal extension of $K(\mathrm{End}\, A_a)$ over which at least one Weil class is defined. Note that by Lemma 5.1.1 we have $K(\mathrm{End}\, \mathsf{A}_a) = K(i) \subseteq K(\zeta_8)$. Theorem 5.2.1 shows that some Weil classes are defined over $K(\zeta_8)$, hence they all are. This implies that the minimal field of definition of all algebraic classes on A_a is $K(\zeta_8)$.

Abdulali [Abd99, Theorem 4.1] shows that, since all Hodge classes on A_a are algebraic, the Hodge conjecture is true for all powers A_a^k of A_a. Since the Mumford–Tate conjecture holds for A_a, by the main theorem of Commelin [Com19] the same is true for A_a^k for all $k \geq 1$. As a consequence, there is a canonical identification between Hodge classes and Tate classes on any power of A_a. Moreover, as above, Moonen [Moo17, Proposition 2.3.2] implies that the Tate conjecture holds for all powers of A_a.

We claim that all cohomology classes of algebraic cycles on all powers of A_a are defined over $K(\zeta_8)$. Write as above $H^1(\mathsf{A}_a(\mathbb{C}), \mathbb{C})$ as the direct sum of two $2g$-dimensional subspaces $Z_\sigma, Z_{\bar\sigma}$ on which $\mathrm{Hg}(\mathsf{A}_a)_{\mathbb{C}}$ acts via the standard representation. The proof of Abdulali [Abd99, Theorem 4.1] shows that the Hodge classes in the cohomology of A_a^k are generated by divisors and by $(\bigwedge^g Z_\sigma)^k \oplus (\bigwedge^g Z_{\bar\sigma})^k$. We have already seen that the classes in $\bigwedge^g Z_\sigma, \bigwedge^g Z_{\bar\sigma} \subset H^g(\mathsf{A}_a(\mathbb{C}), \mathbb{C})$ can be represented by cycles defined over $K(\zeta_8)$, so it suffices to show that the same is true for all divisor classes on A_a^k. To see this, recall that divisor classes are precisely the Hodge classes in $H^2(\mathsf{A}_a^k(\mathbb{C}), \mathbb{C}) \simeq \bigwedge^2 \left(H^1(\mathsf{A}_a(\mathbb{C}), \mathbb{C})^{\oplus k} \right)$. This

space is a direct sum of copies of $\bigwedge^2 H^1(\mathsf{A}_a(\mathbb{C}), \mathbb{C}) \simeq H^2(\mathsf{A}_a(\mathbb{C}), \mathbb{C})$ and $H^1(\mathsf{A}_a(\mathbb{C}), \mathbb{C}) \otimes H^1(\mathsf{A}_a(\mathbb{C}), \mathbb{C})$. Hodge classes in $H^2(\mathsf{A}_a(\mathbb{C}), \mathbb{C})$ are in particular Hodge classes on $\mathsf{A}_a(\mathbb{C})$, hence they are defined over $K(\zeta_8)$ by the above (and in fact, even over K, since the class of the polarization of A_a, which generates $\mathcal{A}^2$, is defined over K). Hodge classes in $H^1(\mathsf{A}_a(\mathbb{C}), \mathbb{C}) \otimes H^1(\mathsf{A}_a(\mathbb{C}), \mathbb{C})$ can be identified with endomorphisms of $\mathsf{A}_a(\mathbb{C})$, and these are all defined over $K(i)$. This concludes the proof of the claim.

Since we have already shown the Tate conjecture for all powers of A_a, we are in a position to apply Corollary 2.4.4, which finally proves $K(\varepsilon_{A_a}) = K(\zeta_8)$. $\square$

5.4. A function field computation.
We complete the argument of section 5.2 by proving that $-\hat{f}|_{P_0}$ is a fourth power in the function field $K(P_0)$. We keep all the notation from section 5.2.

As (natural) generators for the function field $K(\mathrm{Sym}^g(P))$ we may take the elementary symmetric functions $e_1, \ldots, e_g$ in the g variables $t_1, \ldots, t_g$. To make the notation more uniform we also set $e_0 = 1$. From the well-known relation between coefficients of a polynomial and symmetric functions of its roots we obtain $e_i = (-1)^i \frac{Z_{g-i}}{Z_0}$, so that the equations defining P_0 inside $\mathrm{Sym}^g(P)$ give

$$(5.4.1) \qquad e_i + e_{g-i} = (-1)^{g/2+i} c_i e_{g/2}$$

upon restriction to $K(P_0)$. Notice in particular that the equation corresponding to $i = 0$ (or equivalently $i = g$) gives $e_g = -1$ (by Remark 5.2.12 we have $c_g = c_0 = 0$). In order to compute $\hat{f}|_{P_0}$ we then need to write $\hat{f}$ as a rational function $h(e_1, \ldots, e_g)$ of the elementary symmetric functions, and then evaluate it at

$$e_{g-i} = -e_i + (-1)^{g/2+i} c_i e_{g/2} \quad \text{for } i = 0, \ldots, \frac{g}{2} - 1.$$

We start by expressing $\hat{f}$ in terms of symmetric functions. Set for simplicity $\delta_j = -\beta_j$. We have

$$(5.4.2)$$
$$\prod_{i=1}^{g}(t_i - \gamma)^{2g+4} \cdot \hat{f} = \prod_{i=1}^{g}\left(t_i(t_i^2-1)^2 \prod_{j=1}^{d}(t_i - \beta_j)^2\left(t_i - \frac{1}{\beta_j}\right)^2\right)$$

$$= e_g \prod_{i=1}^{g}(1+t_i)^2 \cdot \prod_{i=1}^{g}(1-t_i)^2 \cdot \prod_{j=1}^{d}\prod_{i=1}^{g}(t_i + \delta_j)^2\left(t_i + \frac{1}{\delta_j}\right)^2$$

$$= e_g \left(\sum_{i=0}^{g} e_i\right)^2 \cdot \left(\sum_{i=0}^{g}(-1)^i e_i\right)^2 \cdot \prod_{j=1}^{d}\left(\sum_{i=0}^{g}\delta_j^i e_{g-i}\right)^2\left(\sum_{i=0}^{g}\delta_j^{-i} e_{g-i}\right)^2.$$

Notice that, as g is even, $\prod_{i=1}^{g}(t_i - \gamma)^{2g+4}$ is obviously a fourth power. We now evaluate every factor on the right upon pull-back to $K(P_0)$. Clearly e_g is simply -1. Our task is therefore to show that all the remaining factors multiply up to a fourth power. Next we

have

$$\left(\sum_{i=0}^{g} e_i\right) \cdot \left(\sum_{i=0}^{g}(-1)^i e_i\right) = \left(\frac{1}{2}\sum_{i=0}^{g}(e_i + e_{g-i})\right) \cdot \left(\frac{1}{2}\sum_{i=0}^{g}(-1)^i(e_i + e_{g-i})\right)$$

$$= \left(\frac{1}{2}\sum_{i=0}^{g}(-1)^{g/2+i}c_i e_{g/2}\right) \cdot \left(\frac{1}{2}\sum_{i=0}^{g}(-1)^{g/2}c_i e_{g/2}\right)$$

$$= \left(\sum_{i=0}^{g}(-1)^i c_i\right)\left(\sum_{i=0}^{g}c_i\right) \cdot \frac{e_{g/2}^2}{4}$$

$$= q_1(1,1)q_1(-1,1)\frac{e_{g/2}^2}{Z_{g/2}(q_1)^2},$$

where in the last equality we used Lemma 5.2.13. Finally, from the definition of $q_1(X,Y)$ we obtain

$$\left(\sum_{i=0}^{g} e_i\right) \cdot \left(\sum_{i=0}^{g}(-1)^i e_i\right) = \prod_{j=1}^{d}(1-\delta_j^2)(1-\delta_j^{-2}) \cdot \frac{e_{g/2}^2}{Z_{g/2}(q_1)^2}$$

$$= (-1)^d\prod_{j=1}^{d}(\delta_j - \delta_j^{-1})^2 \cdot \frac{e_{g/2}^2}{Z_{g/2}(q_1)^2}.$$

Finally we consider

$$\left(\sum_{i=0}^{g}\delta_j^i e_{g-i}\right)\left(\sum_{i=0}^{g}\delta_j^{-i} e_{g-i}\right) =$$

$$= \left(\sum_{i=0}^{d}(\delta_j^i e_{g-i} + \delta_j^{g-i}e_i) + \delta_j^{g/2}e_{g/2}\right)\left(\sum_{i=0}^{d}(\delta_j^{-i} e_{g-i} + \delta_j^{i-g}e_i) + \delta_j^{-g/2}e_{g/2}\right).$$

Replacing e_{g-i} by $-e_i + (-1)^{g/2+i}c_i e_{g/2}$ and multiplying the first (resp. second) term by $\delta_j^{-g/2}$ (resp. $\delta_j^{g/2}$) we obtain

$$\left(\sum_{i=0}^{g}\delta_j^i e_{g-i}\right)\left(\sum_{i=0}^{g}\delta_j^{-i} e_{g-i}\right) =$$

$$= \left(\sum_{i=0}^{d}e_i(\delta_j^{g/2-i} - \delta_j^{i-g/2}) + e_{g/2} + (-1)^{g/2}e_{g/2}\sum_{i=0}^{d}\delta_j^{i-g/2}(-1)^i c_i\right) \cdot$$

$$\left(\sum_{i=0}^{d}e_i(\delta_j^{i-g/2} - \delta_j^{g/2-i}) + e_{g/2} + (-1)^{g/2}e_{g/2}\sum_{i=0}^{d}\delta_j^{g/2-i}(-1)^i c_i\right).$$

We will show that the two factors in this last expression are negative of each other, so that their product is minus a square. It is clear that

$$\sum_{i=0}^{d} e_i(\delta_j^{g/2-i} - \delta_j^{i-g/2}) = -\sum_{i=0}^{d} e_i(\delta_j^{i-g/2} - \delta_j^{g/2-i}).$$

We now claim that

$$(5.4.3) \quad e_{g/2} + (-1)^{g/2} e_{g/2} \sum_{i=0}^{d} \delta_j^{i-g/2}(-1)^i c_i = -\left(e_{g/2} + (-1)^{g/2} e_{g/2} \sum_{i=0}^{d} \delta_j^{g/2-i}(-1)^i c_i \right).$$

To show this, we begin by factoring out $e_{g/2}$, multiplying both sides by $(-1)^{g/2}\delta_j^{g/2}$, and bringing everything to the left hand side: our claim can then be rewritten as

$$2(-1)^{g/2}\delta_j^{g/2} + \sum_{i=0}^{d} \delta_j^i(-1)^i c_i + \sum_{i=0}^{d} \delta_j^{g-i}(-1)^i c_i = 0.$$

Recalling that $c_{g-i} = c_i$ and $c_{g/2} = 2$, and using Lemma 5.2.13 again, the above expression may be rewritten as

$$\sum_{i=0}^{g} c_i(-\delta_j)^i = \frac{2}{Z_{g/2}(q_1)} q_1(-\delta_j, 1) = \frac{2}{Z_{g/2}(q_1)} q_1(\beta_j, 1) = 0,$$

as claimed. Inserting this information in our previous computations we obtain

$$\left(\sum_{i=0}^{g} \delta_j^i e_{g-i} \right) \left(\sum_{i=0}^{g} \delta_j^{-i} e_{g-i} \right) =$$

$$= -\left(\sum_{i=0}^{d} e_i(\delta_j^{g/2-i} - \delta_j^{i-g/2}) + e_{g/2} + (-1)^{g/2} e_{g/2} \sum_{i=0}^{d} \delta_j^{i-g/2}(-1)^i c_i \right)^2.$$

Using (5.4.3) again, we further rewrite this as

$$(5.4.4) \quad -\left(\sum_{i=0}^{d} e_i \frac{\delta_j^{g/2-i} - \delta_j^{i-g/2}}{\delta_j - \delta_j^{-1}} + \frac{1}{2}(-1)^{g/2} e_{g/2} \sum_{i=0}^{d} (-1)^i c_i \frac{\delta_j^{i-g/2} - \delta_j^{g/2-i}}{\delta_j - \delta_j^{-1}} \right)^2 (\delta_j - \delta_j^{-1})^2.$$

Recall the well-known fact that for every $k \geq 0$ the expression $\frac{\delta_j^k - \delta_j^{-k}}{\delta_j - \delta_j^{-1}}$ is a polynomial in $(\delta_j + \delta_j^{-1})$. Notice that the coefficients of this polynomial are independent of j, so we write the first factor in (5.4.4) as $\tilde{Q}(\delta_j + \delta_j^{-1})$, where $\tilde{Q}$ is a polynomial in $K(P_0)[x]$. It follows that the product

$$Q := \prod_{j=1}^{g} (-1) \left(\sum_{i=0}^{d} e_i \frac{\delta_j^{g/2-i} - \delta_j^{i-g/2}}{\delta_j - \delta_j^{-1}} + \frac{1}{2}(-1)^{g/2} e_{g/2} \sum_{i=0}^{d} (-1)^i c_i \frac{\delta_j^{i-g/2} - \delta_j^{g/2-i}}{\delta_j - \delta_j^{-1}} \right)$$

$$= \prod_{j} \tilde{Q}(\delta_j + \delta_j^{-1})$$

is a symmetric function of the variables $\{\delta_1 + \delta_1^{-1}, \ldots, \delta_g + \delta_g^{-1}\}$, and therefore, it is a rational function (with coefficients in $K(e_1, \ldots, e_d) = K(P_0)$) of the coefficients of the polynomial $\prod_{j=1}^{g}(t + \delta_j)(t + \delta_j^{-1})$. But this polynomial is in $K[t]$ by definition, so $Q \in K(P_0)$. Putting everything together, (5.4.2) gives

$$\prod_{i=1}^{g}(t_i - \gamma)^{2g+4} \cdot \hat{f}|_{P_0} = -\frac{e_{g/2}^4}{Z_{g/2}(q_1)^4} Q^4 \prod_{j=1}^{d}(\delta_j - \delta_j^{-1})^8.$$

It just remains to notice that

$$R := \prod_{j=1}^{d}(\delta_j - \delta_j^{-1})^2$$

is invariant under $\delta_j \leftrightarrow \delta_j^{-1}$ for all j and under all permutations of the indices, hence lies in K. Therefore $\prod_{j=1}^{d}(\delta_j - \delta_j^{-1})^8 = R^4$ is a fourth power in $K(P_0)$, as claimed.

6. Proof of main result

6.1. **Generic monodromy: overview and setup.** Recall from Setup 3.2.5 the family $\mathsf{C}^{(g)} \to U$, its generic fiber $C^{(g)}$ over $\mathbb{Q}(a)$, and the Jacobian $\mathsf{A}^{(g)} = \operatorname{Jac} C^{(g)}$. (Notice that $\mathbb{Q}(a)$ can be considered as a finitely-generated subfield of $\mathbb{C}$, so all the constructions and considerations of section 2.2 apply.)

Recalling Definition 2.2.9, our main goal in this section is to prove the following theorem.

Theorem 6.1.1. *For every even integer $g \geq 4$ the geometric endomorphism algebra of $A^{(g)}$ is $(-1, -1 \,|\, \mathbb{Q})$ and $A^{(g)}$ is fully of Lefschetz type.*

We will conclude the main result Theorem 1.2.4 from this theorem, as follows. The proof of Theorem 6.1.1 is by induction, but in the inductive step we will rely on Theorem 1.2.4 for genera $g - 2$ and $g - 4$. The reasoning is not circular: we will show below (section 6.5) that Theorem 6.1.1 for genus g implies Theorem 1.2.4 for genus g, see Figure 3. Finally, in Lemma 6.5.2 we will also show Theorem 6.1.1 for $g = 4, 6$ without relying on Theorem 1.2.4, which will complete the inductive proof of both results.
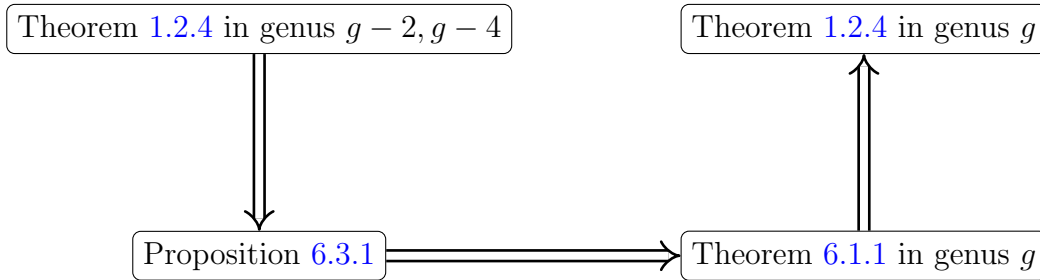

FIGURE 3. Flow of the proof of Theorem 1.2.4

Before embarking, we discuss specializations of abelian schemes (and their endomorphisms) in our geometric setting. Let K be a number field, let U be a normal integral scheme

over K, let $\mathcal{A} \to U$ be an abelian scheme of relative dimension $g \geq 1$, and let $\overline{\eta}$ be the geometric generic point of U. There is a representation $\rho_{\mathcal{A}}$ of the étale fundamental group $\pi_1(U, \overline{\eta})$ with values in $\mathrm{Aut}(\varprojlim_m \mathcal{A}_{\overline{\eta}}[m]) \simeq \mathrm{GL}_{2g}(\widehat{\mathbb{Z}})$ [Zyw19, Introduction]. For a prime ℓ, we may also construct an ℓ-adic representation of the fundamental group

$$(6.1.2) \qquad \rho_{\mathcal{A}, \ell} \colon \pi_1(U, \overline{\eta}) \to \mathrm{Aut}\left(\varprojlim_m \mathcal{A}_{\overline{\eta}}[\ell^m]\right) \simeq \mathrm{GL}_{2g}(\mathbb{Z}_\ell).$$

The two constructions are compatible in an obvious sense: $\rho_{\mathcal{A}, \ell}$ is the composition of $\rho_{\mathcal{A}}$ with the natural projection $\mathrm{GL}_{2g}(\widehat{\mathbb{Z}}) \to \mathrm{GL}_{2g}(\mathbb{Z}_\ell)$. For every $u \in U(K)$ there is a homomorphism $u_* \colon \mathrm{Gal}_K = \pi_1(\mathrm{Spec}\, u, \mathrm{Spec}\, \overline{u}) \to \pi_1(U, \overline{\eta})$, well-defined up to conjugacy, such that the adelic Galois representation attached to the abelian variety $\mathcal{A}_u$ may be identified (up to conjugacy) with the composition $\rho_{\mathcal{A}} \circ u_*$. Similarly, we have $\rho_{\mathcal{A}, \ell} \circ u_* = \rho_{\mathcal{A}_u, \ell}$.

Lemma 6.1.3. *The image of the adelic representation $\pi_1(U, \overline{\eta}) \to \mathrm{Aut}(\varprojlim_m \mathcal{A}_{\overline{\eta}}[\ell^m])$ attached to the abelian scheme $\mathcal{A} \to U$ coincides with the image of the adelic representation $\rho_A \colon \mathrm{Gal}_{K(U)} \to \mathrm{Aut}(\varprojlim_m \mathcal{A}_{\overline{\eta}}[\ell^m])$ attached to the abelian variety $A/K(U)$, where A is the generic fiber of $\mathcal{A}$.*

Proof. Let $\eta = \mathrm{Spec}\, K(U)$ be the generic point of U and $\overline{\eta}$ be the corresponding geometric generic point. By [Sta22, Proposition 0BQM], the natural map $\eta_* \colon \pi_1(\eta, \overline{\eta}) \to \pi_1(U, \overline{\eta})$ is surjective. The claim follows from the formula $\rho_A = \rho_{\mathcal{A}} \circ \eta_*$. $\qquad\square$

In particular, one may view the image of the Galois representation attached to $\mathcal{A}_u$ as a subgroup of $\rho_{\mathcal{A}}(\pi_1(U, \overline{\eta}))$. When U is an open subscheme of $\mathbb{P}^n_K$, for most $u \in U(K)$ this inclusion has small index, as made precise by the following result.

Theorem 6.1.4 (Zywina [Zyw19, Theorem 1.1])**.** *Let U be an open subscheme of $\mathbb{P}^n_K$. Let H be the absolute multiplicative height on $\mathbb{P}^n(K)$. Then there exists a constant c such that*

$$(6.1.5) \qquad \lim_{x \to \infty} \frac{\#\{u \in U(K) : H(u) \leq x, [\rho_{\mathcal{A}}(\pi_1(U, \overline{\eta})) : \rho_{\mathcal{A}_u}(\mathrm{Gal}_K)] \leq c\}}{\#\{u \in U(K) : H(u) \leq x\}} = 1.$$

Suppose now that $U \subseteq \mathbb{A}^d_{\mathbb{Q}}$ is a nonempty open subscheme. Write A for the generic fiber of $\mathcal{A}$ (over $\mathbb{Q}(U)$). Consider further an integral, closed, normal subscheme $T \subseteq U$. Let $\mathcal{A}_T := \mathcal{A} \times_U T$ and its special fibre A_T over the function field of T. We collect in the next lemma some basic properties of the specialization homomorphism.

Lemma 6.1.6. *The following statements hold.*

  (a) *The endomorphism ring of $\mathcal{A}$ coincides with the endomorphism ring of its generic fiber A.*
  (b) *The endomorphism ring of $\mathcal{A}$ injects into the endomorphism ring of $\mathcal{A}_T$.*
  (c) *The image of the ℓ-adic monodromy representation attached to $\mathcal{A}_T$ (resp. A_T) is contained in the image of the ℓ-adic monodromy representation attached to $\mathcal{A}$ (resp. A).*
  (d) *The endomorphism ring of A (resp. A^{al}) injects into the endomorphism ring of A_T (resp. A_T^{al}).*

Proof. For (a), it suffices to show that any endomorphism of $\mathcal{A}$ that is defined over a nonempty open subscheme V of U extends to all of U, and this follows immediately from work of Faltings [Fal83, Lemma 1].

For (b), we apply the rigidity lemma [MFK, Corollary 6.2] (in the notation there, take $X = G = A$, $f = \phi$, and $g = 0$).

For (c), we start with the case of $\mathcal{A}_T$ and $\mathcal{A}$. Notice that the given embedding $T \hookrightarrow U$ induces a map of fundamental groups $\pi_1(T, \overline{\eta_T}) \to \pi_1(U, \overline{\eta})$, well-defined up to conjugation (which takes care of the arbitrary choice of the base point $\overline{\eta_T}$), such that the following diagram commutes:

$$
\begin{array}{ccc}
\pi_1(T, \overline{\eta_T}) & \longrightarrow & \pi_1(U, \overline{\eta_U}) \\
{\scriptstyle \rho_{\mathcal{A}_T, \ell}} \downarrow & & \downarrow {\scriptstyle \rho_{\mathcal{A}, \ell}} \\
\operatorname{Aut}(T_\ell \mathcal{A}_T) & \longrightarrow & \operatorname{Aut}(T_\ell \mathcal{A})
\end{array}
$$

The claim follows immediately.

The statement for A_T and A then follows from Lemma 6.1.3, since the images of the Galois representations are the same for the scheme and the generic fiber.

Finally, we prove (d). The statement concerning A and A_T is an immediate consequence of (a) and (b). Let now L be the subfield of $\mathbb{Q}(U)^{\mathrm{al}}$ over which all endomorphisms of A are defined. Letting U_L and T_L be the normalizations of U, T in L, we can apply (a) and (b) to $\mathcal{A}_{U_L}$ and $\mathcal{A}_{T_L}$ to get injectivity at the level of the geometric endomorphism rings. $\qquad\square$

Suppose furthermore that T is an open subscheme of $\mathbb{A}^1_{\mathbb{Q}}$ (in particular, T is normal). Then $\mathcal{A}_T$ admits a Néron model $\mathfrak{A}$ over all of $\mathbb{A}^1_{\mathbb{Q}}$, and we can consider the special fibres of $\mathfrak{A}$ even at the points of $\mathbb{A}^1_{\mathbb{Q}}$ where $\mathcal{A}_T$ has bad reduction. In this more specific situation we have the following lemma.

Lemma 6.1.7. *Let $y \in \mathbb{A}^1(\mathbb{Q})$ be such that $\mathfrak{A}_y$ is an abelian variety.*

(a) *The endomorphism ring of A_T (resp. A_T^{al}) injects into the endomorphism ring of the special fibre $\mathfrak{A}_y$ of $\mathfrak{A}$ at y (resp. $\mathfrak{A}_y^{\mathrm{al}}$) .*

(b) *Let $y \in \mathbb{A}^1(\mathbb{Q})$ and suppose that the fibre $\mathfrak{A}_y$ is an abelian variety. The image of the ℓ-adic Galois representation attached to $\mathfrak{A}_y$ injects into the image of the ℓ-adic Galois representation attached to $\mathcal{A}_T$.*

Proof. By [Fal83, Lemma 1] the endomorphism ring of $\mathcal{A}_T$ is the endomorphism ring of A_T. The universal property of Néron models implies that endomorphisms of $\mathcal{A}_T$ extend to $\mathfrak{A}$, so that $\operatorname{End}(A_T) = \operatorname{End}(\mathcal{A}_T) = \operatorname{End}(\mathfrak{A})$.

We now prove the desired injectivity. Let R be a Dedekind domain containing $\mathbb{Q}$, $F = \operatorname{Frac}(R)$, A an abelian variety over F, and $\mathfrak{A}$ its Néron model over R. Let y be a closed point of $\operatorname{Spec} R$ and let $\kappa(y)$ be the residue field at y. Suppose that $\mathfrak{A}_y$ is an abelian variety. The specialization map $A(F) = \mathfrak{A}(R) \to \mathfrak{A}_y(\kappa(y))$ is injective on torsion points by [BLR90, Proposition 3 in §7.3] (notice that all positive integers are invertible in R). We will prove in general that $\operatorname{End}(\mathfrak{A}) \to \operatorname{End}(\mathfrak{A}_y)$ is injective. Note that the universal property of Néron models shows $\operatorname{End}(\mathfrak{A}) = \operatorname{End}(A)$. To prove injectivity, take an element φ in the kernel. Let

$P \in A(F^{\mathrm{al}})_{\mathrm{tors}}$ and let L be a finite extension of F over which P is defined. Denote by R_L the integral closure of R in L and by $\mathfrak{A}_L$ the Néron model of A_L. Fix a point y_L of $\mathrm{Spec}\, R_L$ over y. Since $\mathfrak{A}_y$ is an abelian variety, $(\mathfrak{A}_L)_{y_L}$ is simply the base-change of $\mathfrak{A}_y$ to $\kappa(y_L)$. In particular, φ induces zero also on $(\mathfrak{A}_L)_{y_L}$. Hence, the image of $\varphi(P)$ in $(\mathfrak{A}_L)(\kappa(y_L))$ is zero, but $\varphi(P) \in \mathfrak{A}_L(L) = A(L)$ is torsion, and specialization is injective on torsion points, so $\varphi(P) = 0$. Since this holds for all torsion points P, the endomorphism φ is zero, as claimed in part (a). The statement about geometric endomorphism rings is proved exactly in the same way.

For part (b), arguing as in part (a) and passing to the limit over all L (equivalently, over all torsion points), we get canonical identifications $A(K^{\mathrm{al}})_{\mathrm{tors}} \simeq (\mathfrak{A}_y)(\kappa(y)^{\mathrm{al}})_{\mathrm{tors}}$. These identifications are clearly Galois-equivariant, and the claim follows. □

Lemmas 6.1.6 and 6.1.7 imply the following.

Proposition 6.1.8. *Let U be an open subscheme of $\mathbb{A}_{\mathbb{Q}}^d$, and let $\mathcal{A}$ be an abelian scheme over U, with generic fibre A over $\mathbb{Q}(U)$. Let T be a closed subscheme of U isomorphic to an open subscheme of $\mathbb{A}_{\mathbb{Q}}^1$. Let $\mathfrak{A}/\mathbb{A}_{\mathbb{Q}}^1$ be the Néron model for the restriction $\mathcal{A}_T$ of $\mathcal{A}$ to T. For every $x \in \mathbb{A}_{\mathbb{Q}}^1(\mathbb{Q})$ such that the special fibre $\mathfrak{A}_x/\mathbb{Q}$ is an abelian variety, there are injective specialization maps $\mathrm{End}(A) \to \mathrm{End}(\mathfrak{A}_x)$ and $\mathrm{End}(A^{\mathrm{al}}) \to \mathrm{End}(\mathfrak{A}_x^{\mathrm{al}})$. The image of the ℓ-adic Galois representation attached to $\mathcal{A}$ contains the image of the ℓ-adic Galois representation attached to $\mathfrak{A}_x$.*

Remark 6.1.9. Decompositions up to isogeny of an abelian variety A correspond to idempotents in the endomorphism algebra of A. As a consequence, whenever endomorphism specialize injectively (such as in the situation of the previous proposition), splittings of abelian varieties also specialize.

6.2. **Generically geometrically simple: base cases.** Recall the abelian variety $A^{(g)}$ from the beginning of section 6.1. Over the next two sections, we show that $A^{(g)}$ is geometrically simple. We start by taking care of the special cases $g = 4$ and $g = 6$.

Lemma 6.2.1. *Let $g \in \{4, 6\}$. Then the abelian variety $A^{(g)}$ has geometric endomorphism algebra $(-1, -1 \mid \mathbb{Q})$. In particular, $A^{(g)}$ is geometrically simple.*

Proof. By (3.2.10), we have an injective map $(-1, -1 \mid \mathbb{Q}) \hookrightarrow \mathrm{End}(A^{(g),\mathrm{al}})_{\mathbb{Q}}$. On the other hand, by Theorem 4.1.1 for $g = 4$ and Proposition 4.1.8 for $g = 6$, there exist specializations $\mathsf{A}_a^{(g)}$ with $(\mathrm{End}\, \mathsf{A}_a^{(g)})_{\mathbb{Q}} = (-1, -1 \mid \mathbb{Q})$. By Proposition 6.1.8 and a dimension count, we conclude that $\mathrm{End}(A^{(g),\mathrm{al}})_{\mathbb{Q}} = (-1, -1 \mid \mathbb{Q})$. □

In our inductive argument, we will use a degeneration (specialization) argument. Since the argument is a bit notationally heavy, we briefly work with the special case $g = 4$ to introduce some of the key ideas; we use this also in section 7.2.

Proposition 6.2.2. *Under the map $\mathbb{Q}[b] \to \mathbb{Q}[t]$ by $b \mapsto t^2$, the Néron model of the pullback of $\mathsf{B}^{(4)} = \mathrm{Jac}\, \mathsf{D}^{(4)}$ is an abelian scheme over $\mathbb{P}^1$ in the variable t.*

Proof. The essential information necessary to understand the reduction properties of a hyperelliptic curve are contained in its *cluster picture* [DDMM22, BBB$^+$22], observing that the results extend to the situation of a DVR with perfect infinite residue field.

Our family is smooth away from $t^2 = 0, \pm 1, \infty, \pm i$, so (following the notation in the above articles) we look over the complete local rings $\mathcal{O}_K$ at these points, with K the field of fractions of $\mathcal{O}_K$. For example, at $t = 0$, we have $\mathcal{O}_K = \mathbb{Q}[[t]]$ and $K = \mathbb{Q}((t))$. The cluster picture organizes the set of roots $\mathcal{R} = \{0, \pm 1, \pm i, \pm t^2, \pm 1/t^2\}$ according to discs they belong to.

For example, at $t = 0$, we have the proper clusters

$$(6.2.3) \qquad \{0, \pm t^2, \pm 1, \pm i, \pm 1/t^2\} \supset \{0, \pm t^2, \pm 1, \pm i\} \supset \{0, \pm t^2\}$$

of depths $-2, 0, 2$. The criterion for good reduction of the Jacobian [BBB$^+$22, Theorem 5.4] has three parts:

   (1) the set of roots $\mathcal{R}$ is defined over $K(i)$, which is unramified over $K = \mathbb{Q}((b))$;
   (2) all (proper) clusters $\mathfrak{s} \neq \mathcal{R}$ are odd; and
   (3) all proper clusters $\mathfrak{s}$ are principal with $v_{\mathfrak{s}} \in 2\mathbb{Z}$, since the leading coefficient $1/t^4$ has even valuation and all depths are even.

(In particular, repeating the calculation in (3) for the family over $\mathbb{Q}[b]$ we see it has bad, potentially good, reduction: two clusters have odd depth.)

Although it is not necessary to prove the lemma, we also obtain a description of the special fiber as coming from maximal principal odd subclusters: we obtain the equations (over $\mathbb{Q}^{\mathrm{al}} \simeq \mathbb{Q}^{\mathrm{al}}[[t]]/(t)$)

$$(6.2.4) \qquad \begin{aligned} y^2 &= x(x^2 - 1) \\ y^2 &= -x(x^4 - 1) \\ y^2 &= x(x^2 - 1) \end{aligned}$$

for each of the three clusters, respectively, attached from top to bottom. We do a similar calculation, but in a self-contained way, with explicit equations in Lemma 6.3.6.

We repeat this at the other valuations. For $t = \infty$, we get a result symmetric with $t = 0$. For $t^2 = 1$ we have the clusters

$$\{0, \pm t^2, \pm 1, \pm i, \pm 1/t^2\} \supset \{-1, -1/t^2, -t^2\}, \{1, 1/t^2, t^2\}$$

with depths 0 and $2, 2$; the same criterion applies, giving $y^2 = x(x^4 - 1)$ with two elliptic curves $y^2 = -x(x^2 - 1)$ attached. A symmetric argument applies for $t^2 = -1, \pm i$. $\square$

6.3. Generically geometrically simple: induction.
We now make our key inductive argument.

Proposition 6.3.1. *Let $g \geq 8$ be even. Suppose Theorem 1.2.4 holds in genera $g - 2$ and $g - 4$. Then $A^{(g)}$ has geometric endomorphism algebra $(-1, -1 \mid \mathbb{Q})$.*

Recall from (3.3.4) the family $\mathsf{D}^{(g)}$ of nice curves over V with Jacobian $\mathsf{B}^{(g)} := \operatorname{Jac} \mathsf{D}^{(g)}$. The generic fiber $B^{(g)} = \operatorname{Jac} D^{(g)}$ of $\mathsf{B}^{(g)}$ is the base change of $A^{(g)}$ under $\mathbb{Q}(a) \hookrightarrow \mathbb{Q}(b)$.

Thus it suffices to show that $B^{(g)}$ is geometrically simple with geometric endomorphism algebra $(-1, -1 \,|\, \mathbb{Q})$.

Note that $B^{(g)}$ is an abelian scheme over an open subscheme of $\mathbb{A}_{\mathbb{Q}}^d$, so we can apply Proposition 6.1.8 to $B^{(g)}$. Adapting the notation of that proposition, an abelian variety of the form $\mathfrak{B}B_x$ will be called a *specialization* of $B^{(g)}$.

We will prove Proposition 6.3.1 by constructing two specializations of $B^{(g)}$ whose structure is sufficiently different to force the geometric endomorphism algebra of $B^{(g)}$ to be $(-1, -1 \,|\, \mathbb{Q})$. These specializations are constructed in the next two lemmas.

Lemma 6.3.2. *Let $g \geq 8$ be even and suppose that Theorem 1.2.4 holds in genus $g - 4$. Let $c \in \mathbb{Q}^\times \backslash \{\pm 1\}$. Then there is a specialization of $B^{(g)}$ whose decomposition up to isogeny over $\mathbb{Q}^{\mathrm{al}}$ is given by $B^{(g),\mathrm{al}} \sim X \times (E_c)^4$, where*

- *X over $\mathbb{Q}^{\mathrm{al}}$ is simple with $\dim(X) = g - 4$ and $\mathrm{End}(X)_{\mathbb{Q}} \simeq (-1, -1 \,|\, \mathbb{Q})$, and*
- *E_c is the elliptic curve over $\mathbb{Q}^{\mathrm{al}}$ with j-invariant $j(c)$ given by (6.3.5).*

Proof. Fix $\overline{b} := (\overline{b_1}, \ldots, \overline{b_{d-2}}) \in \mathbb{Q}^{d-2}$ such that the Jacobian of the hyperelliptic curve over $\mathbb{Q}$ given by

$$ \mathsf{C}_{\overline{b}}^{(g-4)} : y^2 = x(x^4 - 1) \prod_{n=1}^{d-2} \left(x^2 - \overline{b_n}^2 \right) \left(x^2 - \overline{b_n}^{-2} \right) $$

is geometrically simple with endomorphism ring $(-1, -1 \,|\, \mathbb{Q})$. Such $\overline{b_i}$ exist by Theorem 1.2.4 in genus $g - 4$, which holds by assumption. To ease the notation, we will denote $\mathsf{C}_{\overline{b}}^{(g-4)}$ simply by $\Gamma^{(g-4)}$.

Further fix $c \in \mathbb{Q}^\times \backslash \{\pm 1\}$. Consider now the family $\mathcal{X}_c$ of nice curves given by the Cartesian diagram

$$ \begin{array}{ccc} \mathcal{X}_c & \longrightarrow & \mathsf{D}^{(g)} \\ \downarrow & & \downarrow \\ \mathbb{A}_{\mathbb{Q}}^1 \backslash \Delta' & \longrightarrow & V, \end{array} $$

where the bottom arrow sends t to $(\overline{b_1}, \ldots, \overline{b_{d-2}}, t^2, ct^2)$, and Δ' is a suitable proper closed subscheme of $\mathbb{A}_{\mathbb{Q}}^1$ such that $\mathcal{X}_c$ is smooth over its complement. Concretely, $\mathcal{X}_c$ is the hyperelliptic curve with equation

$$ y^2 = x(x^4 - 1)(x^2 - t^4)(x^2 - t^{-4})(x^2 - (ct^2)^2)(x^2 - (ct^2)^{-2}) \prod_{n=1}^{d-2} \left(x^2 - \overline{b_n}^2 \right) \left(x^2 - \overline{b_n}^{-2} \right), $$

where we have identified $\mathbb{A}_{\mathbb{Q}}^1$ with $\mathrm{Spec}\,\mathbb{Q}[t]$.

Let $\mathsf{A}_{\mathcal{X}_c}$ be the Jacobian of $\mathcal{X}_c$, and let $\mathfrak{A}_{\mathcal{X}_c}$ be its Néron model over the Dedekind domain $\mathbb{Q}[t]$. We claim that the special fibre of $\mathfrak{A}_{\mathcal{X}_c}$ at $t = 0$ is the product of $\mathrm{Jac}(\Gamma^{(g-4)})$ by $A_c^+ \times A_c^-$, where $A_c^{\pm}$ is the Jacobian of the genus 2 hyperelliptic curve with equation $y^2 = \pm z(z^2 - 1)(z^2 - c^2)$. To see this, we compute the special fibre of the stable model $\mathcal{Z}_c$

of $\mathcal{X}_c$ at $t = 0$. Over the open locus where t is invertible (that is, the complement of the closed subscheme defined by t), the curve $\mathcal{X}_c$ is isomorphic to

$$\mathcal{X}_c' : Y^2 = x(x^4-1)(x^2-t^4)(t^4x^2-1)(x^2-(ct^2)^2)(t^4x^2-c^{-2})\prod_{n=1}^{d-2}\left(x^2-\overline{b_n}^2\right)\left(x^2-\overline{b_n}^{-2}\right),$$

with $Y = t^4 y$. The curve $\mathcal{X}_c'$ clearly extends over the closed subscheme defined by (t), and its special fibre at $t = 0$ is given by

$$\overline{\Gamma^{(g-4)}} : Y^2 = c^{-2}x^4x(x^4-1)\prod_{n=1}^{d-2}\left(x^2-\overline{b_n}^2\right)\left(x^2-\overline{b_n}^{-2}\right).$$

Reabsorbing the square factor $c^{-2}x^4$ into Y^2, we see that the normalisation of this curve is simply $\Gamma^{(g-4)}$. To understand the other components of the special fibre of the stable model we blow-up the special fibre at $x = 0$, $y = 0$. Writing $x = t^2 z$ and $y = tv/c$ we obtain
(6.3.3)
$$\mathcal{X}_c' : (v/c)^2 = z(t^8z^2-1)(z^2-1)(t^8z^2-1)(z^2-c^2)(t^8z^2-c^{-2})\prod_{n=1}^{d-2}\left(t^4z^2-\overline{b_n}^2\right)\left(t^4z^2-\overline{b_n}^{-2}\right).$$

This maps to the blow-up at the origin of $\mathbb{A}^1_{\mathbb{Q}[t]}$ along (t^2). Let E be the exceptional divisor of the blow-up of $\mathbb{A}^1_{\mathbb{Q}[t]}$ at 0; the function z is naturally a coordinate on E. Specialising the above equation at $t = 0$, we obtain the equation of a double cover of E: it is the genus-2 hyperelliptic curve given explicitly by $v^2 = -z(z^2-1)(z^2-c^2)$. It meets $\overline{\Gamma^{(g-4)}}$ at the point $x = 0$ of the latter.

The automorphism $(x, y) \mapsto (1/x, iy/x^{g+1})$ of the family restricts to the special fibre of the stable model, and acts on $\overline{\Gamma^{(g-4)}}$ by sending $x \mapsto 1/x$. In particular, this automorphism sends the genus-2 component at $x = 0$ to an isomorphic one at $x = \infty$. Note that the isomorphism is only defined over $\mathbb{Q}(i)$, since this holds for the automorphism (the same result can also be obtained by a direct calculation completely analogous to the above, which also gives the equation for the component at $x = \infty$). Thus, the stable model $\mathcal{Z}_c$ of the special fibre of $\mathcal{X}_c'$ is given by the union of the three curves $\overline{\Gamma^{(g-4)}}$ (with normalisation $\Gamma^{(g-4)}$), $v^2 = -z(z^2-1)(z^2-c^2)$ and $v^2 = z(z^2-1)(z^2-c^{-2})$, meeting as in Section 6.3.

As a consequence (see [BLR90, Example 8 on p. 246]), the special fibre of the Néron model $\mathfrak{A}_{\mathsf{C}_x}$ at $t = 0$ is the Jacobian of the (reducible) stable curve $\mathcal{Z}_c$, hence it is isomorphic to the product of the Jacobians of the curves in Section 6.3. We denote by A_c^+, A_c^- the Jacobians of the two genus-2 components. Notice that A_c^+, A_c^- are isomorphic over $\mathbb{Q}(i)$.

We claim that $A_c^\pm$ is isogenous over $\mathbb{Q}^{\mathrm{al}}$ to $E_c^+ \times E_c^-$, with $E_c^\pm$ the elliptic curve

$$(6.3.4) \qquad\qquad E_c^\pm : w^2 = (u \pm 2\sqrt{c})(u^2-(c+1)^2).$$

To prove this it is enough to notice that there are maps

$$v^2 = -z(z^2-1)(z^2-c^2) \rightarrow w^2 = (u \pm 2\sqrt{c})(u^2-(c+1)^2)$$

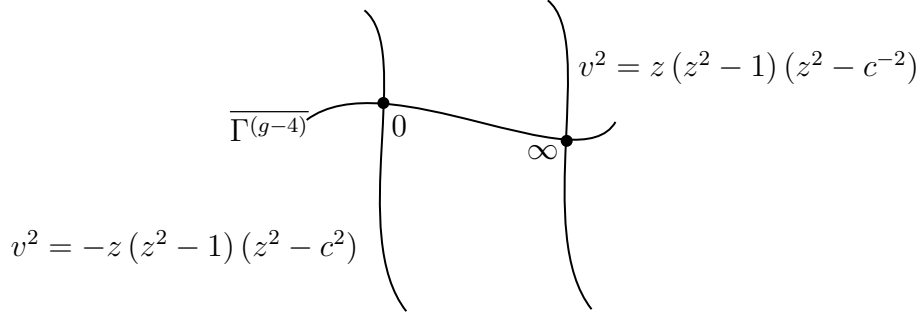$$v^2 = z(z^2 - 1)(z^2 - c^{-2})$$

$$v^2 = -z(z^2 - 1)(z^2 - c^2)$$

FIGURE 4. The special fibre of the stable model

given by

$$u = z + \frac{c}{z}, \quad w = iv\frac{z \pm \sqrt{c}}{z^2}.$$

The j-invariant of each curve $E_c^{\pm}$ is given by the non-constant rational function in c

$$(6.3.5) \qquad j(E_c^{\pm}) = 1728\frac{(c + 1/3)^3(c + 3)^3}{(c - 1)^4(c + 1)^2}.$$

In particular, E_c^+ and E_c^- are geometrically isomorphic. The lemma follows upon taking $E = E_c^{+,\mathrm{al}} \simeq E_c^{-,\mathrm{al}}$ and $X = \mathrm{Jac}((\Gamma^{(g-4)})^{\mathrm{al}})$. $\qquad\square$

We now look for a different specialization whose geometric endomorphism algebra is sufficiently different from what we just found.

Lemma 6.3.6. *Let $g \geq 6$ be even and assume that Theorem 1.2.4 holds in genus $g - 2$. Then the following statements hold.*

(a) *There is a specialization $Z/\mathbb{Q}$ of $B^{(g)}$ isomorphic to $Y \times E \times E'$, where Y is a geometrically simple abelian variety over $\mathbb{Q}$ of dimension $g - 2$, with $\mathrm{End}(Y^{\mathrm{al}})_{\mathbb{Q}} \simeq (-1, -1 \mid \mathbb{Q})$ and $G_{Y,\ell}^0$ isomorphic to a form of $\mathbb{G}_m \cdot \mathrm{SO}_{g-2}$ for every prime ℓ, and E, E' are elliptic curves with j-invariant 1728.*

(b) *There exists an injection of $\mathrm{End}(B^{(g),\mathrm{al}})_{\mathbb{Q}}$ into $(-1, -1 \mid \mathbb{Q}) \times \mathrm{Mat}_{2\times2}(\mathbb{Q}(i))$ which sends $[-1] \in \mathrm{End}(B^{(g),\mathrm{al}})_0$ to $(-1, -1) \in (-1, -1 \mid \mathbb{Q}) \times \mathrm{Mat}_{2\times2}(\mathbb{Q}(i))$.*

(c) *The abelian variety $B^{(g),\mathrm{al}}$ has a simple isogeny factor of dimension at least $g - 2$.*

Proof. The proof is similar to that of Lemma 6.3.2, so we give fewer details. Fix $\overline{\beta} := (\overline{\beta_1}, \ldots, \overline{\beta_{d-1}}) \in \mathbb{Q}^{d-1}$ such that the Jacobian $Y/\mathbb{Q}$ of

$$\mathsf{C}_{\overline{\beta}}^{(g-2)} : y^2 = x(x^4 - 1)\prod_{n=1}^{d-1}\left(x^2 - \overline{\beta_n}^2\right)\left(x^2 - \overline{\beta_n}^{-2}\right)$$

is geometrically simple, has geometric endomorphism ring $(-1, -1 \mid \mathbb{Q})$ and monodromy group $G_{Y,\ell}$ isomorphic to a form of $\mathbb{G}_m \cdot \mathrm{SO}_{g-2}$ for every prime ℓ. The existence of $\overline{\beta}$ is

guaranteed by Theorem 1.2.4 in genus $g - 2$, which we have assumed. As in the proof of Lemma 6.3.2, to ease the notation we write $\Gamma^{(g-2)}$ for $\mathsf{C}^{(g-2)}_{\overline{\beta}}$.

Proceeding as in that proof, consider the family $\mathcal{Y}$ of nice curves of genus $g - 2$ given by the Cartesian diagram

$$
\begin{array}{ccc}
\mathcal{Y} & \longrightarrow & \mathsf{D}^{(g)} \\
\downarrow & & \downarrow \\
\mathbb{A}^1_{\mathbb{Q}} \setminus \Delta'' & \longrightarrow & V,
\end{array}
$$

where the bottom arrow sends t to $(\overline{\beta_1}, \ldots, \overline{\beta_{d-1}}, t^2)$, and Δ'' is a suitable proper closed subscheme of $\mathbb{A}^1_{\mathbb{Q}}$ such that $\mathcal{Y}$ is smooth over its complement. Denote by $\mathfrak{A}_{\mathcal{Y}}$ the Néron model over $\mathbb{A}^1_{\mathbb{Q}}$ of the Jacobian of $\mathcal{Y}$. By a calculation completely analogous to that in the proof of Lemma 6.3.2, the special fibre at $t = 0$ of $\mathcal{J}_{\mathcal{Y}}$ is geometrically the product of the simple abelian variety $Y^{\mathrm{al}} = \mathrm{Jac}(\Gamma^{(g-2)})^{\mathrm{al}}$ and E^2, where $E/\mathbb{Q}^{\mathrm{al}}$ is the elliptic curve $y^2 = x^3 - x$. Notice that $\mathrm{Jac}(\Gamma^{(g-2)})^{\mathrm{al}}$, being simple of dimension $g - 2 > 1$, has no common isogeny factors with E^2. The desired injection

$$
\mathrm{End}(B^{(g),\mathrm{al}})_{\mathbb{Q}} \hookrightarrow \mathrm{End}(\mathrm{Jac}(\Gamma^{(g-2)})^{\mathrm{al}} \times E^2)_{\mathbb{Q}} \simeq (-1, -1 \,|\, \mathbb{Q}) \times \mathrm{Mat}_{2 \times 2}(\mathbb{Q}(i))
$$

is obtained by specialization (Proposition 6.1.8), and takes multiplication by $[-1]$ to $(-1, -\mathrm{Id})$. Moreover, since any (geometric) splitting of $B^{(g)}$ would be reflected by a splitting of the geometric special fibre of $\mathfrak{A}_{\mathcal{Y}}$ at $t = 0$ (see Remark 6.1.9), this implies that $B^{(g),\mathrm{al}}$ has a simple isogeny factor of dimension at least $g - 2$. $\qquad\square$

We are now ready to prove Proposition 6.3.1.

Proof of Proposition 6.3.1. By Lemma 6.3.6, $B^{(g),\mathrm{al}}$ has a simple isogeny factor of dimension at least $g - 2$:

(6.3.7) $$ B^{(g),\mathrm{al}} \sim X \times Y $$

where X is simple over $\mathbb{Q}(b)^{\mathrm{al}}$ with $\dim X = g - 2, g - 1, g$ (and $\dim(X) + \dim(Y) = g$, but Y may not be simple). For dimension reasons ($g \geq 8$) and since X is simple, there is no isogeny factor in common between X and Y, so

$$ \mathrm{End}(B^{(g),\mathrm{al}})_{\mathbb{Q}} \simeq \mathrm{End}(X)_{\mathbb{Q}} \times \mathrm{End}(Y)_{\mathbb{Q}}. $$

Since $\mathcal{O} \subset (-1, -1 \,|\, \mathbb{Q}) \hookrightarrow (\mathrm{End}\, B^{(g),\mathrm{al}})_{\mathbb{Q}}$ and since $(-1, -1 \,|\, \mathbb{Q})$ acts nontrivially on every subspace in its tangent representation (see the proof of Lemma 3.1.1), there is an embedding of $(-1, -1 \,|\, \mathbb{Q})$ in both $\mathrm{End}(X)_{\mathbb{Q}}$ and in $\mathrm{End}(Y)_{\mathbb{Q}}$. There is no nontrivial action of $(-1, -1 \,|\, \mathbb{Q})$ on an elliptic curve in characteristic 0, so to conclude that $B^{(g)}$ is geometrically simple, we assume for purposes of contradiction that $\dim(X) = g - 2$ in (6.3.7). Again, we have $(-1, -1 \,|\, \mathbb{Q}) \hookrightarrow (\mathrm{End}\, Y)_{\mathbb{Q}}$; but over an algebraically closed field of characteristic 0, the only abelian surface Y (up to isogeny) that admits such an action is $Y \sim E^2$, where E is the elliptic curve $y^2 = x^3 - x$. Thus, E is a (geometric) isogeny factor of any specialization of $B^{(g)}$ (see Remark 6.1.9, and notice that E can only specialize to E, since it is already defined over $\mathbb{Q}$).

However, for each $c \in \mathbb{Q}^\times \setminus \{\pm 1\}$, Lemma 6.3.2 gives a specialization of $B^{(g)}$ which is geometrically isogenous to $X \times E_c^4$, with X (geometrically) simple of dimension $g - 4 > 1$ and where E_c has j-invariant given by (6.3.5). Choosing c in such a way that E_c is not geometrically isogenous to E we obtain a contradiction. Such c are easily seen to exist: for example, $c = 2$ will do, because $j(E_2)$ is not an integer by (6.3.5), so E_2 does not have potential complex multiplication and is therefore not (potentially) isogenous to E. This proves as desired that $B^{(g)}$ is geometrically simple.

Knowing this, we conclude the proof as follows. We have $\mathbb{Q}$-algebra maps

$$\operatorname{End}(B^{(g),\mathrm{al}})_\mathbb{Q} \hookrightarrow (-1, -1 \,|\, \mathbb{Q}) \times \mathrm{M}_2(\mathbb{Q}(i)) \to (-1, -1 \,|\, \mathbb{Q});$$

the first map comes from the specialization in Lemma 6.3.6(b), the second by projection onto the first factor. The composition of these maps is nonzero, since Lemma 6.3.6(b) provides that $[-1] \mapsto (-1, -1) \mapsto -1$. Since $\operatorname{End}(B^{(g),\mathrm{al}})_\mathbb{Q}$ is a simple $\mathbb{Q}$-algebra, this composition is injective. But by construction $(-1, -1 \,|\, \mathbb{Q}) \hookrightarrow \operatorname{End}(B^{(g),\mathrm{al}})_\mathbb{Q}$, so by dimensions this must be an isomorphism. $\qquad \square$

6.4. **Proof of generic monodromy.** We now have all the ingredients to prove the main result of this section, Theorem 6.1.1, conditionally on Theorem 1.2.4 holding for genera $g - 2$ and $g - 4$. We will also prove Theorem 6.1.1 for $g = 4, 6$ without relying on Theorem 1.2.4.

Proposition 6.4.1. *If the statement of Theorem 1.2.4 holds for $g - 2, g - 4$, then Theorem 6.1.1 holds for g.*

Proof. The first part follows from Lemma 6.2.1 (for $g \leq 6$) and Proposition 6.3.1 (for $g \geq 8$). Consider now the specialization $Z = Y \times E \times E'$ of $\mathsf{B}^{(g)}$ given by Lemma 6.3.6. Since E, E' have the same j-invariant, there is a finite extension F of $\mathbb{Q}$ such that $Z_F \simeq Y_F \times E^2$. By Proposition 6.1.8, the image of the ℓ-adic Galois representation attached to Z_F is contained in the image of the ℓ-adic Galois representation attached to $B^{(g)}$ (hence the same holds for the Zariski closures). By [Lom16b, Lemma 3.4], since E is a CM elliptic curve, the rank of the ℓ-adic monodromy group of $Z_F \simeq Y_F \times E^2$ is equal to

$$\big(\operatorname{rank}(G_{Y,\ell}^0) - 1\big) + \big(\operatorname{rank}(G_{E,\ell}^0) - 1\big) + 1,$$

where the summands ± 1 take into account the torus of homotheties. By Lemma 6.3.6(a) we have $\operatorname{rank}(G_{Y,\ell}^0) = \operatorname{rank}(\mathbb{G}_m \cdot \mathrm{SO}_{g-2}) = 1 + \frac{g-2}{2}$, and since E is CM it is well-known that $\operatorname{rank} G_{E,\ell}^0 = 2$, so we obtain that the rank of $G_{B^{(g)},\ell}^0$ is at least $g/2 + 1$. On the other hand, by Lemma 2.3.2 we know that $L(B^{(g)})$ has rank $g/2$. Thus $G_{B^{(g)},\ell}^0$ and $(L(B^{(g)}) \cdot \mathbb{G}_m)_{\mathbb{Q}_\ell}$ are connected reductive groups of the same rank, with the former contained in the latter, and they have the same centralizer (in both cases this is $\operatorname{End}(B^{(g),\mathrm{al}})_{\mathbb{Q}_\ell} = (-1, -1 \,|\, \mathbb{Q}) \otimes_\mathbb{Q} \mathbb{Q}_\ell$: for $G_{B^{(g)},\ell}^0$ this follows from Faltings's proof of Tate's conjecture, for $L(B^{(g)}) \cdot \mathbb{G}_m$ it follows from the definition and the double centralizer theorem). By [Win02, Lemma 7], this implies $G_{B^{(g)},\ell}^0 = (L(B^{(g)}) \cdot \mathbb{G}_m)_{\mathbb{Q}_\ell}$, and since $G_{B^{(g)},\ell}^0 \subseteq \mathrm{MT}(B^{(g)})_{\mathbb{Q}_\ell} \subseteq (L(B^{(g)}) \cdot \mathbb{G}_m)_{\mathbb{Q}_\ell}$ we obtain the final statement of the theorem, i.e., $B^{(g)}$ (and hence $A^{(g)}$) is fully of Lefschetz type. $\qquad \square$

6.5. **Variation of the Galois image in the family A_a.** In this section we conclude the inductive step by showing that the truth of Theorem 6.1.1 for a certain genus g implies the truth of Theorem 1.2.4 for the same genus. We apply Theorem 6.1.4 to the abelian scheme $A^{(g)} \to U$. Let c be the corresponding constant. The set

$$X_g = \{a \in U(\mathbb{Q}) : [\rho_{A^{(g)}}(\pi_1(U, \overline{\eta})) : \rho_{A_a^{(g)}}(\mathrm{Gal}_{\mathbb{Q}})] \leq c\}$$

has full density inside $U(\mathbb{Q})$ in the sense of (6.1.5).

Lemma 6.5.1. *Suppose that Theorem 6.1.1 holds in genus g. Then for every $a \in X_g$ we have the following:*

(a) $\mathrm{End}(A_a^{\mathrm{al}})_{\mathbb{Q}} = (-1, -1 \,|\, \mathbb{Q})$;
(b) $\mathrm{Hg}(A_a)_{\mathbb{C}} \cong \mathrm{SO}_{g, \mathbb{C}}$; *and*
(c) A_a *satisfies the Mumford–Tate conjecture.*

Proof. It is easy to see that, given two subgroups $G_1 \subseteq G_2 \subseteq \mathrm{GL}_{2g}(\mathbb{Q}_\ell)$ with $[G_2 : G_1] < \infty$, the Zariski closures of G_1, G_2 in $\mathrm{GL}_{2g, \mathbb{Q}_\ell}$ have the same dimension (consider their Lie algebras). As a consequence, the defining property of X_g shows that, for every $a \in X_g$ and every prime ℓ, the ℓ-adic monodromy group of A_a has the same dimension as the ℓ-adic monodromy group of the generic fibre $A^{(g)}$. In particular, both groups have the same connected component of the identity. By Tate's conjecture on endomorphisms for abelian varieties over finitely generated fields of characteristic 0 (proved by Faltings [Fal84]), the connected component of the ℓ-adic monodromy group determines the dimension of the geometric endomorphism algebra. More explicitly, if A is an abelian variety over a field K of characteristic 0 and ℓ is a prime number, we have

$$\mathrm{End}(A^{\mathrm{al}})_{\mathbb{Q}_\ell} \simeq \mathrm{End}_{G_{A,\ell}^0}(T_\ell A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

It follows from the above that $\dim_{\mathbb{Q}} \mathrm{End}(A_a^{\mathrm{al}})_{\mathbb{Q}} = \dim_{\mathbb{Q}_\ell} \mathrm{End}(A_a^{\mathrm{al}})_{\mathbb{Q}_\ell} = \dim_{\mathbb{Q}_\ell} \mathrm{End}(A^{(g),\mathrm{al}})_{\mathbb{Q}_\ell} = 4$, where the last equality follows from the fact that $\mathrm{End}(A^{(g),\mathrm{al}})_{\mathbb{Q}} \simeq (-1, -1 \,|\, \mathbb{Q})$ by Theorem 6.1.1, which we are assuming. By (3.2.10) there is an injection of $(-1, -1 \,|\, \mathbb{Q})$ in $\mathrm{End}(A_a^{\mathrm{al}})_{\mathbb{Q}}$, so we obtain the desired equality $\mathrm{End}(A_a^{\mathrm{al}})_{\mathbb{Q}} = (-1, -1 \,|\, \mathbb{Q})$, that is, (a).

Now observe that A_a has the same polarization and geometric endomorphism algebra, and therefore the same Lefschetz group, as the generic fibre $A^{(g)}$. By definition of the set X_g, the image $\rho_{A_a, \ell}(\mathrm{Gal}_{\mathbb{Q}})$ has finite index inside $\rho_{A^{(g)}, \ell}(\pi_1(U, \overline{\eta})) = \rho_{A^{(g)}, \ell}(\mathrm{Gal}_{\mathbb{Q}(a)})$ (see Lemma 6.1.3 for the last equality). By Theorem 6.1.1, $A^{(g)}$ satisfies the Mumford–Tate conjecture, so $\rho_{A^{(g)}, \ell}(\mathrm{Gal}_{\mathbb{Q}(a)})$ has finite index inside the $\mathbb{Q}_\ell$-points of $L(A^{(g)}) \cdot \mathbb{G}_m = L(A_a) \cdot \mathbb{G}_m$. By Theorem 2.2.7(a)–(b) we also have the containments

$$\rho_{A_a, \ell}(\mathrm{Gal}_{\mathbb{Q}}) \subseteq \mathrm{MT}(A_a)(\mathbb{Q}_\ell) \subseteq (L(A_a) \cdot \mathbb{G}_m)(\mathbb{Q}_\ell).$$

It follows from the above discussion that these inclusions all have finite index. In particular, $\mathrm{MT}(A_a)$ and $L(A_a) \cdot \mathbb{G}_m$ have the same dimension, so the inclusion $\mathrm{Hg}(A_a) \subseteq L(A_a)$ is an equality. Since $L(A_a)_{\mathbb{C}} = L(A^{(g)})_{\mathbb{C}}$ is isomorphic to $\mathrm{SO}_{g, \mathbb{C}}$ by Lemma 2.3.2, this proves (b).

Finally, the Zariski closure of $\rho_{A_a, \ell}(\mathrm{Gal}_{\mathbb{Q}})$ is $\mathrm{MT}(A_a)(\mathbb{Q}_\ell)$ (the former is open in the latter even for the ℓ-adic topology, since it has finite index), so the Mumford–Tate conjecture

holds at the prime ℓ. By [LP95, Theorem 4.3], this implies that Mumford–Tate holds for all primes, that is, (c). $\qquad\square$

We now give a proof of Theorem 6.1.1 for $g = 4, 6$ that does not rely on Theorem 1.2.4: this will establish the base cases of our induction to show Theorem 1.2.4.

Lemma 6.5.2. *Theorem 6.1.1 holds for $g = 4, 6$.*

Proof. The first statement in Theorem 6.1.1 follows from Lemma 6.2.1. An abelian variety of dimension 4 or 6 with geometric endomorphism algebra $(-1, -1 \mid \mathbb{Q})$ is fully of Lefschetz type, see [MZ95] for $g = 4$, and [BGK10] for $g = 6$. $\qquad\square$

We are finally ready to prove Theorem 1.2.4.

Proof of Theorem 1.2.4. The proof is by induction, see Figure 3. We have shown in section 6.1 that Theorem 1.2.4 in genera $g - 2$ and $g - 4$ implies Theorem 6.1.1 in genus g. Since Lemma 6.5.2 takes care of the base cases $g = 4$ and $g = 6$ of Theorem 6.1.1, it suffices to show that Theorem 6.1.1 in genus g implies Theorem 1.2.4 in genus g.

From Theorem 6.1.4, Lemma 6.5.1, and the definition of the set X_g it follows that for a density 1 subset of points $a \in U(\mathbb{Q})$ when ordered by height all of the following hold: the curve $\mathsf{C}_a^{(g)}$ is smooth, a_d is nonzero, the Jacobian $\mathsf{A}_a^{(g)}$ has geometric endomorphism algebra $(-1, -1 \mid \mathbb{Q})$, satisfies the Mumford–Tate conjecture, and has complex Hodge group isomorphic to $\mathrm{SO}_{g,\mathbb{C}}$. For any such a, properties (i) and (ii) in Theorem 1.2.4 follow from Lemma 5.1.1, (iii) and (iv) follow from Theorem 5.3.1, and (v) is clear. Finally, the last part of the statement is a consequence of Corollary 6.6.3 below. $\qquad\square$

6.6. **Infinitely many isogeny classes.** We conclude by showing the final statement in Theorem 1.2.4.

Proposition 6.6.1. *Let S be a variety defined over a number field K, and let $\mathsf{C} \to S$ be a family of smooth, projective, geometrically connected curves of genus g. Suppose that the induced moduli map $S(\overline{K}) \to M_g(K^{\mathrm{al}})$ has finite fibres. The map*

$$
\begin{array}{rcl}
S(K) & \to & \{abelian\ varieties\ over\ K^{\mathrm{al}}\}/ \sim \\
s & \mapsto & \mathrm{Jac}(\mathsf{C}_s)^{\mathrm{al}},
\end{array}
$$

where $\sim$ denotes isogeny over K^{al}, has finite fibres. In particular, if $S(K)$ is an infinite set, the abelian varieties $\{\mathrm{Jac}(\mathsf{C}_s) : s \in S(K)\}$ fall into infinitely many distinct isogeny classes over K^{al}.

Proof. For the sake of simplicity write A_s for $\mathrm{Jac}(\mathsf{C}_s)$. It is an abelian variety over K. By contradiction, let $s_0 \in K$ be such that $\mathcal{T}(s_0) := \{s \in S(K) : A_s \sim_{K^{\mathrm{al}}} A_{s_0}\}$ is an infinite set. By [Sil92], every homomorphism between A_s and A_{s_0} is defined over the field $F_s := K(A_s[3], A_{s_0}[3])$, whose degree over $\mathbb{Q}$ is bounded independently of s_0 and s. Hence, for all $s \in \mathcal{T}(s_0)$, the abelian varieties A_s and A_{s_0} are isogenous over F_s, and by the explicit isogeny theorem [GR14, Théorème 1.4], there exists an F_s-isogeny $A_{s_0} \to A_s$ of degree at most $b(h(A_{s_0}), [F_s : \mathbb{Q}], g)$, where $b(h(A_{s_0}), g, [F_s : \mathbb{Q}])$ depends only on the stable Faltings height $h(A_{s_0})$ of A_{s_0}, on the dimension g, and on the degree $[F_s : \mathbb{Q}]$. Since $h(A_{s_0})$, g and $[F_s : \mathbb{Q}]$

are bounded independently of $s \in \mathcal{T}(s_0)$, we obtain that there exists an integer $N = N(s_0)$ such that for every $s \in \mathcal{T}(s_0)$ there exists a K^{al}-isogeny $A_{s_0} \to A_s$ of degree at most N. Since A_{s_0} has only finitely many subgroups of order at most N, the infinitely many abelian varieties A_s^{al} for $s \in \mathcal{T}(s_0)$ fall into finitely many isomorphism classes. Since every abelian variety carries at most finitely many isomorphism classes of principal polarizations [NN81], we see that infinitely many $s \in \mathcal{T}(s_0)$ give rise to (geometrically) isomorphic *principally polarized* abelian varieties A_s. Let $\mathcal{T}'(s_0)$ be an infinite subset of $\mathcal{T}(s_0)$ for which $A_s, A_{s'}$ are geometrically isomorphic principally polarized abelian varieties for all $s, s' \in \mathcal{T}'(s_0)$. By Torelli's theorem, the curves C_s are then geometrically isomorphic for all $s \in \mathcal{T}'(s_0)$. This contradicts the assumption that the moduli map $S(K^{\text{al}}) \to M_g(K^{\text{al}})$ has finite fibres and finishes the proof of the proposition. $\qquad\square$

Remark 6.6.2. For a given $s_0 \in S(K)$ one could ask for a bound on the cardinality of the set

$$\{s \in S(K) : \mathrm{Jac}(\mathsf{C}_s) \sim_{K^{\text{al}}} \mathrm{Jac}(\mathsf{C}_{s_0})\}.$$

Following the proof above, one would need to bound the number of (isomorphism classes of) principal polarizations on the abelian varieties $\mathrm{Jac}(\mathsf{C}_s)$. As shown in [Lan87], there is no uniform bound on this quantity depending only on the genus of C, and in fact, Rotger [Rot03] proved that for every positive integer N there are N non-isomorphic genus-2 curves with isomorphic unpolarized Jacobian.

The following corollary implies the last statement in Theorem 1.2.4.

Corollary 6.6.3. *As $a = (a_1, \ldots, a_d)$ varies in $U(\mathbb{Q})$, the Jacobians A_a over $\mathbb{Q}$ of the curves C_a fall into infinitely many distinct isogeny classes over $\mathbb{Q}^{\text{al}}$.*

Proof. Follows from Proposition 6.6.1 and Lemma 3.4.1(b) (finite-to-one). $\qquad\square$

Remark 6.6.4. Let $g > 4$ and $d = g/2 - 1$. We sketch a different proof that the Jacobians A_a fall into infinitely many distinct isogeny classes over $\mathbb{Q}^{\text{al}}$. It is straightforward to see that for every prime p sufficiently large relative to d there are distinct rational numbers $b_1, \ldots, b_d$ different from $0, \pm 1$ such that $v_p(b_1 - b_2) > 0$, but $v_p(b_1 - r) = 0$ for all roots $r \neq b_1, b_2$ of the separable polynomial defining (3.3.4). We take $a = (a_1, \ldots, a_d) = \phi(b_1, \ldots, b_d)$ as in section 3.3 (just expanding out the polynomial). Then by the criterion given by the cluster picture [BBB+22, Theorem 5.5], the Jacobian A_a does *not* have potentially good reduction at p: the cluster $\{x \in \overline{\mathbb{Q}_p} : v_p(x - b_1) \geq 1\}$ is even.

It follows that for all sufficiently large primes p, there exists a specialization A_a that does not have potentially good reduction at p. Since the set of places of potential good reduction is a $\mathbb{Q}^{\text{al}}$-isogeny invariant, the Jacobians A_a must belong to infinitely many distinct $\mathbb{Q}^{\text{al}}$-isogeny classes. (When $g = 4$, an analogue of the previous argument is prevented by Lemma 4.4.2.)

7. Moduli interpretation and final remarks

We conclude by relating the family $\mathsf{C}^{(4)}$ of Setup 3.2.5 to moduli: we define a moduli space of abelian varieties in section 7.1 and compare it to our family in section 7.2. We conclude in section 7.3 by a complex analytic description.

7.1. **Moduli space.** We begin by defining a moduli space of abelian varieties. We follow Lan [Lan13, §1.2], including his notation.

As above, let $B := \left(\dfrac{-1, -1}{\mathbb{Q}} \right)$ and let $\mathcal{O} := \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}ij \subset B$ be the Lipschitz order. Let $L \simeq \mathbb{Z}^8$ be the (left) $\mathcal{O}$-lattice defined by the action in (4.2.2); equip L with the symplectic (nondegenerate) pairing $\langle\, ,\, \rangle$ defined by (4.2.3) and the polarization defined by $h = \alpha^*$. The tuple $(L, \langle\, ,\, \rangle, h)$ is called a *PEL-type $\mathcal{O}$-lattice* (and elsewhere, an *integral PEL datum*, taking the standard involution as the positive involution of $\mathcal{O}$). As Lan elaborates [Lan23, §5.1.1–5.1.2], a PEL-type $\mathcal{O}$-lattice yields a complex abelian variety with PEL (polarization, endomorphism, and level) structure: in our case,

- the abelian variety $A_0 := (L \otimes_{\mathbb{Z}} \mathbb{R})/L$ whose complex structure is given by h,
- the polarization $\lambda_0 \colon A_0 \to (L \otimes_{\mathbb{Z}} \mathbb{R})/L^{\#}$ where $L^{\#} \simeq L$ is the dual under $\langle\, ,\, \rangle$,
- endomorphisms $\mathcal{O} \to \operatorname{End}(A_0)$, and
- full level 2-structure given by $(\frac{1}{2}L)/L \xrightarrow{\sim} A_0[2]$.

Moving from one object to the general case, we define an associated moduli problem $\mathcal{M}_2(L, \langle\, ,\, \rangle, h)$ following Lan [Lan13, Definition 1.4.1.2] as follows (taking for $\square$ the set of odd primes). Let $S_0 := \operatorname{Spec}(\mathbb{Z}[1/2])$. Define the category fibered in groupoids over the category Sch_{S_0} of schemes over S_0, whose fiber over each scheme S is the groupoid $\mathcal{M}_2(L, \langle\, ,\, \rangle, h)(S)$ described as follows. The objects of $\mathcal{M}_2(L, \langle\, ,\, \rangle, h)(S)$ are tuples $(A, \lambda, \iota, \alpha_2)$, where:

(1) A is an abelian scheme over S;
(2) $\lambda \colon A \to A^{\vee}$ is a $\mathbb{Z}[1/2]^{\times}$-polarization of A [Lan13, Definition 1.3.2.19];
(3) $\iota \colon \mathcal{O} \to \operatorname{End}_S(A)$ defines an $\mathcal{O}$-structure of (A, λ);
(4) $\underline{\operatorname{Lie}}_{A/S}$ with its $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$-module structure given naturally by ι satisfies the Kottwitz determinantal condition [Lan13, Definition 1.3.4.1] given by $(L \otimes_{\mathbb{Z}} \mathbb{R}, \langle\, ,\, \rangle, h)$;
(5) $\alpha_n \colon (L/2L)_S \xrightarrow{\sim} A[2]$ is an (integral) principal level-2 structure of (A, λ, ι) of type $L \otimes_{\mathbb{Z}} \mathbb{Z}_2, \langle\, ,\, \rangle$.

The notion of isomorphism [Lan13, Definition 1.4.1.2] is the expected one, so is omitted. (The reason for the polarization structure as it is given in (2) is a bit technical, see [Lan13, §1.3.1].)

Theorem 7.1.1. *The moduli problem $\mathcal{M}_2(L, \langle\, ,\, \rangle, h)$ is an algebraic stack which is separated, smooth, and of finite type over S_0.*

Proof. The result follows from [Lan13, Corollary 1.4.1.12] once we show that the reflex field of $\mathcal{M}_2(L, \langle\, ,\, \rangle, h)$ is equal to $F_0 = \mathbb{Q}$.

As in Lan [Lan13, (1.2.5.1)], we have a decomposition $L \otimes_{\mathbb{Z}} \mathbb{C} \simeq V_0 \oplus \overline{V_0}$ for a certain $\mathbb{C}$-submodule V_0 of dimension equal to $\frac{1}{2} \operatorname{rk}_{\mathbb{Z}} L = 4$. We check the definition [Lan13, Definition 1.2.5.4], and show that for every $\sigma \in \operatorname{Aut}(\mathbb{C} \mid \mathbb{Q})$, we have an isomorphism $V_0 \simeq V_0 \otimes_{\mathbb{C}, \sigma} \mathbb{C}$ of $B \otimes_{\mathbb{Q}} \mathbb{C}$-modules. Indeed, as $B \otimes_{\mathbb{Q}} \mathbb{C} \simeq \operatorname{M}_2(\mathbb{C})$, every $B \otimes_{\mathbb{Q}} \mathbb{C}$-module is a direct sum of copies of the standard module $\mathbb{C}^2$. Since V_0 and $V_0 \otimes_{\mathbb{C}, \sigma} \mathbb{C}$ have the same dimension over $\mathbb{C}$, they are both isomorphic to $(\mathbb{C}^2)^{(\dim V_0)/2} = (\mathbb{C}^2)^2$. $\square$

Remark 7.1.2. Moduli stacks of abelian varieties with PEL structure are also Shimura stacks (see Deligne [Del71] and Milne [Mil05, §8]): for the explicit description of the Mumford–Tate datum (G, X), we refer also to Abdulali [Abd99, §4].

7.2. **Explicit recognition.** We now recall section 3.3, in particular the family of nice curves defined by (3.3.1). We saw in Proposition 6.2.2 that under the pullback provided by $t = b^2$, that the Néron model we obtain an abelian scheme of dimension 4 over $\mathbb{P}^1$ (in the parameter t). Away from $t^2 = b = 0, \pm 1, \infty, \pm i$, the fibers are smooth, principally polarized, equipped with endomorphisms by $\mathcal{O}$ defined over $\mathbb{Q}(i)$ and full level 2-torsion; these extend to the entire Néron model, which gives a point of $\mathcal{M}_2(L, \langle\,,\,\rangle, h)(\mathbb{P}^1_{\mathbb{Q}(i)})$ and so defines a morphism

$$(7.2.1) \qquad \varsigma \colon \mathbb{P}^1_{\mathbb{Q}(i)} \to \mathcal{M}_2(L, \langle\,,\,\rangle, h)$$

Let $\mathscr{M}$ be the image of ς. For properties of stacky curves, we refer to Voight–Zureick-Brown [VZB22, Chapter 5].

Theorem 7.2.2. *The map ς in (7.2.1) has degree 2, and its image is a stacky curve with coarse space $\mathbb{P}^1$ and two stacky points with nontrivial stabilizer μ_2.*

Proof. We recall the calculation in the proof of Lemma 3.4.1(c). Here it is simpler, as the curve is still defined up to isomorphism by the parameter b (giving a degree 2 map), and on each smooth fiber we have marked the 10 Weierstrass points, so the only possible automorphism is ± 1; but these automorphisms do not change the parameter $b = t^2$. Thus ς has degree 2.

The image is a stacky curve whose coarse space [VZB22, Proposition 5.3.3] (due to Keel–Mori [KM97]) is a smooth curve receiving a map from $\mathbb{P}^1$, so it is isomorphic to $\mathbb{P}^1$. The map ς is therefore induced by $\mathbb{P}^1_{\mathbb{Q}(i)} \mapsto [\mathbb{P}^1_{\mathbb{Q}(i)}/\mu_2]$ where μ_2 is acting by $t \mapsto -t$, which introduces μ_2-stabilizers exactly at $0, \infty$. (We can also see this from the computation of the stable model (Proposition 6.2.2): although the map $t \mapsto -t$ acts trivially over smooth fibers, in the stable limit it still acts trivially on the genus 2 component but acts by the hyperelliptic involution on the two genus 1 components.) $\square$

Remark 7.2.3. One could extend the above description to arbitrary genus $g \geq 6$, and one would obtain a map from our family of curves U to an associated moduli stack $\mathcal{M}$ of abelian varieties with PEL structure. However, the dimension of the image $U \to \mathcal{M}_g \hookrightarrow \mathcal{A}_g$ is $d = g/2 - 1$ by Lemma 3.4.1(b), whereas the dimension of $\mathcal{M}$ is $(g/2 - 1)(g/4)$; so only for $g = 4$ do these dimensions match. Somewhat surprisingly, we find that for $g = 4$, the moduli space $\mathcal{M}$ lies *inside* the Torelli locus! For general (even) $g \geq 4$, the expected dimension of the intersection between the Torelli locus and $\mathcal{M}$ is

$$(g/2 - 1)(g/4) + 3g - 3 - g(g+1)/2 = \frac{3}{2}(2 - g/2)(g/2 - 1).$$

Already for $g = 4$, this gives expected dimension 0, so we have an unlikely intersection. For $g = 6$, the expected dimension is -3, but our family has dimension 2. We do not have an explanation for this curious phenomenon.

7.3. Complex uniformization.
To conclude, we relate our family to the analytic (complex) uniformization, following the notation in Birkenhake–Lange [BL04, §9.5].

First we note that that the dimension of the moduli space of principally polarized complex abelian fourfolds equipped with definite quaternion multiplication by $\mathcal{O}$ (and a fixed polarization) is 1 [BL04, p. 261] (the dimension is $(e/2)m(m-1) = 1$ where $e = 1$ and $m = 2$).

Fix an $(-1, -1 \,|\, \mathbb{Q})$-equivariant isomorphism of $\mathbb{R}$-vector spaces $\varphi : (-1, -1 \,|\, \mathbb{Q})_{\mathbb{R}}^2 \to \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. For example, let $\varphi \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e_1 \otimes 1$ and $\varphi \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e_5 \otimes 1$, where $e_1, \ldots, e_8$ is the basis of Λ given by the columns of the period matrix Π from section 4.3.

We consider the pullback to $(-1, -1 \,|\, \mathbb{Q})_{\mathbb{R}}^2$ of the (imaginary part of the) polarization on $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, that is, the intersection form given in (4.2.3). There exists $T \in \mathrm{Mat}_2((-1, -1 \,|\, \mathbb{Q})_{\mathbb{R}})$ such that
$$\mathrm{trd}(^t\underline{a}T\underline{b}') = (\mathrm{Im}\, H)(\varphi(\underline{a}), \varphi(\underline{b})) \quad \forall \underline{a}, \underline{b} \in (-1, -1 \,|\, \mathbb{Q})_{\mathbb{R}}^2,$$
where $\mathrm{Im}\, H$ is the anti-symmetric bilinear form on $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ corresponding to the imaginary part of the polarization, and for a vector $\underline{a} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in (-1, -1 \,|\, \mathbb{Q})_{\mathbb{R}}^2$ we have set $\underline{a}' = \begin{pmatrix} \overline{h_1} \\ \overline{h_2} \end{pmatrix}$.
For every $q \in (-1, -1 \,|\, \mathbb{Q})_{\mathbb{R}}$ we have
$$2q = \mathrm{trd}(q) + \mathrm{trd}(-qi)i + \mathrm{trd}(-qj)j + \mathrm{trd}(-qk)k;$$
applying this identity to the $(1, 1)$-coefficient of T, denoted by t_{11}, and using the fact that we have $\mathrm{trd}(^t\underline{a}T\underline{b}') = (\mathrm{Im}\, H)(\varphi(\underline{a}), \varphi(\underline{b}))$, we obtain
$$2t_{11} = (\mathrm{Im}\, H)\left(\varphi \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \varphi \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) + (\mathrm{Im}\, H)\left(\varphi \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \varphi \begin{pmatrix} i \\ 0 \end{pmatrix} \right) i$$
$$+ (\mathrm{Im}\, H)\left(\varphi \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \varphi \begin{pmatrix} j \\ 0 \end{pmatrix} \right) j + (\mathrm{Im}\, H)\left(\varphi \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \varphi \begin{pmatrix} k \\ 0 \end{pmatrix} \right) k.$$

Using the explicit matrix representations of the imaginary part of the polarization (4.2.3) and of the action of i and j (4.2.2) we obtain $2t_{11} = -(i + j)$. Repeating the same calculation for the other coefficients of T yields
$$T = \frac{1}{2} \begin{pmatrix} -(i + j) & 0 \\ 0 & k \end{pmatrix}.$$

We observe that T satisfies $T' = -T$, as predicted by the general theory, where $'$ denotes the transpose-conjugate matrix. The reduced norm of T is the product of the reduced norms of $-(i + j)/2$ and $k/2$, hence $\mathrm{Nrd}(T) = 1/8$. Notice that this is not a square in $\mathbb{Q}$ [BL04, Exercise 9.10.2 (a)].

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