ON GALOIS INERTIAL TYPES OF ELLIPTIC CURVES OVER $\mathbb{Q}_\ell$

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ABSTRACT. We provide a complete, explicit description of the inertial Weil–Deligne types arising from elliptic curves over $\mathbb{Q}_\ell$ for ℓ prime.

1. Introduction

1.1. **Motivation.** In the classification of finite-dimensional complex representations of the absolute Galois group of a local field, it has proven very useful to classify by restriction to the inertia subgroup [8, 26, 7, 22]. In this article, we will pursue an explicit classification for such representations coming from elliptic curves.

Let ℓ be prime and let $F \supseteq \mathbb{Q}_\ell$ be a finite extension with algebraic closure F^{al}. Let W_F be the Weil group of F, the subgroup of $\mathrm{Gal}(F^{\mathrm{al}} \,|\, F)$ acting by an integer power of the Frobenius map on the maximal unramified subextension. Let $(\rho\colon W_F \to \mathrm{GL}_n(\mathbb{C}), N)$ be an n-dimensional (complex) Weil–Deligne representation (Definition 2.1.2). Let $I_F \leq W_F$ be the inertia subgroup. An inertial (Weil–Deligne) type (also called a Galois inertial type) is a pair (τ, N) where $\tau = \rho|_{I_F}$ for a Weil–Deligne representation (ρ, N). To ease notation, we will often abbreviate the pair (τ, N) by τ (and indeed often we have $N = 0$ anyway).

Already the case $n = 2$ is interesting and rich, and it is this case we will consider here. Inertial types for 2-dimensional representations were introduced by Conrad–Diamond–Taylor [10] and Breuil–Conrad–Diamond–Taylor [4] in the study of deformation rings of Galois representations and were used in the proof of modularity of elliptic curves over $\mathbb{Q}$. Diamond–Kramer [15, Appendix] described the analogously defined type (as in Diamond [14]) of the mod p Galois representation $\overline{\rho}_{E,p}$ attached to an elliptic curve E over F in terms of the j-invariant of E; in particular, they give a description of the restriction of $\overline{\rho}_{E,p}$ to I_F in as much detail as possible using only $j(E)$.

Types have also been studied in the context of Galois representations attached more generally to classical modular forms. For example, in Loeffler–Weinstein [23, 24] an algorithm to determine the restriction to decomposition groups of such representations was described and implemented; this includes a description of the inertial type. By counting the number of inertial types attached to modular forms, Dieulefait–Pacetti–Tsaknias [16] have given a precise generalization of the Maeda conjecture.

Additionally, inertial types have played a prominent role in the mod p and p-adic Langlands program. Henniart [5, Appendix] showed that there is an *inertial Langlands correspondence* between 2-dimensional Galois inertial types of F and smooth representations of $\mathrm{GL}_2(\mathcal{O}_F)$, where $\mathcal{O}_F$ denotes the ring of integers of F. Indeed, the Breuil–Mézard conjecture [5] for $\mathbb{Q}_\ell$ can be seen as a refinement of Serre's conjecture over $\mathbb{Q}$, where inertial types are a crucial

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input. The inertial Langlands correspondence for general $n \geq 2$ was proven by Paskunas [26].

Diophantine applications provide another important motivation to study inertial types for GL_2. In Bennett–Skinner [2], the *image of inertia argument* was introduced and successfully applied to solve certain Fermat equations. Recently, further refinements and applications of this argument were obtained by Billerey–Chen–Dieulefait–Freitas [3]: we may be able to distinguish between the mod p representations attached to elliptic curves over a global field by showing they have different images of inertia [3, Section 3]. Therefore, the more we know about inertial types of elliptic curves, the greater the applicability of this argument. In this direction, Freitas–Naskręcki–Stoll [18, Theorem 3.1] describe the possible fixed fields of the restriction $\overline{\rho}_{E,p}|_{I_{\mathbb{Q}_\ell}}$ to inertia for elliptic curves E over $\mathbb{Q}_\ell$ with certain reduction types at $\ell = 2, 3$, and they applied this to study solutions of the generalized Fermat equation $x^2 + y^3 = z^p$.

In light of these applications, the goal of this paper is to give a complete, explicit description of the inertial types for all elliptic curves E over $\mathbb{Q}_\ell$. Our main theorem (Theorem 1.2 below) has already been applied to the determination of the symplectic type of isomorphisms between the p-torsion of elliptic curve by Freitas–Kraus [19].

1.2. **Main result.** Our main result (combining Propositions 4.1.1, 4.2.1, and 5.2.1 and Theorems 6.1.6 and 7.1.2) is as follows. Let E be an elliptic curve over $F = \mathbb{Q}_\ell$. Attached to E is an inertial Weil–Deligne type τ_E obtained from the action on the (dual of the) p-adic Tate module for a prime $p \neq \ell$, independent of p (for details, see section 3.1). If E has potentially good reduction, then this good reduction is obtained over a minimal field $L \supseteq F$ and we define its semistability defect to be $e := [L : F]$.

Main Theorem. *Let E be an elliptic curve over $\mathbb{Q}_\ell$ with conductor N_E and inertial Weil–Deligne type τ_E; if E has additive, potentially good reduction, let e be its semistability defect. Then τ_E is classified up to equivalence according to Table 1.*

The notation in Table 1 is explained in section 2.5. In Table 1, the exceptional (or primitive) supercuspidal representations are labelled as such and collected in the last rows of the table, whereas the nonexceptional (imprimitive) supercuspidal representations are labelled simply *supercuspidal*, for brevity.

All types in Table 1 arise for an elliptic curve over $\mathbb{Q}_\ell$: see Examples 5.2.2 and 6.1.7.

Our method of proof of our Main Theorem is by direct, exhaustive calculation: we deduce the inertial type associated to an elliptic curve over $\mathbb{Q}_\ell$ in terms of its reduction type. We have endeavored to streamline these calculations while still remaining comprehensive and as self-contained as possible. Indeed, many of our calculations can be found in other places in the literature: for example, the 3-adic types are already implicitly given in the proof of the modularity theorem [4], Dieulefait–Pacetti–Tsaknias [16] more generally identify local invariants of Galois orbits of classical newforms (relevant here for weight $k = 2$). More recently, Coppola [11, 12] has studied wild Galois representations ($\ell = 3$ and $e = 12$; $\ell = 2$ and $e = 8, 24$) over more general local fields.

1.3. **Outline.** The paper is organized as follows. In sections 2–3, we establish background by briefly reviewing some facts concerning 2-dimensional Weil–Deligne representations, inertial types, and elliptic curves. In section 4, we compute types in the case of potentially multiplicative reduction for all primes ℓ and for additive, potentially good reduction for

Reduction type	ℓ	e	$v_\ell(N_E)$	τ_E	Description
good	-	-	0	trivial	trivial
multiplicative	-	-	1	$\tau_{\mathrm{St},\ell}$	special
additive, potentially multiplicative	≥ 3	-	2	$\tau_{\mathrm{St},\ell} \otimes \varepsilon_\ell$	special
	2	-	4	$\tau_{\mathrm{St},2} \otimes \varepsilon_{-4}$	
			6	$\tau_{\mathrm{St},2} \otimes \varepsilon_{\pm 8}$	
additive, potentially good	≥ 5	2	2	ε_ℓ	principal series
		$3, 4, 6 \mid (\ell - 1)$		$\tau_{\mathrm{ps},\ell}(1, 1, e)$	
		$3, 4, 6 \mid (\ell + 1)$		$\tau_{\mathrm{sc},\ell}(u, 2, e)$	supercuspidal
	3	2	2	ε_3	principal series
		3	4	$\tau_{\mathrm{ps},3}(1, 2, 3)$	
		3	4	$\tau_{\mathrm{sc},3}(-1, 2, 3)$	supercuspidal
		4	2	$\tau_{\mathrm{sc},3}(-1, 1, 4)$	
		6	4	$\tau_{\mathrm{ps},3}(1, 2, 3) \otimes \varepsilon_3$	principal series
		6	4	$\tau_{\mathrm{sc},3}(-1, 2, 3) \otimes \varepsilon_3$	supercuspidal
		12	3	$\tau_{\mathrm{sc},3}(\pm 3, 2, 6)$	
			5	$\tau_{\mathrm{sc},3}(-3, 4, 6)_j \quad (j = 0, 1, 2)$	
	2	2	4	ε_{-4}	principal series
			6	$\varepsilon_{\pm 8}$	
		3	2	$\tau_{\mathrm{sc},2}(5, 1, 3)$	supercuspidal
		4	8	$\tau_{\mathrm{ps},2}(1, 4, 4) \otimes \varepsilon_d \quad (d = 1, -4)$	principal series
				$\tau_{\mathrm{sc},2}(5, 4, 4) \otimes \varepsilon_d \quad (d = 1, -4)$	supercuspidal
		6	4	$\tau_{\mathrm{sc},2}(5, 1, 3) \otimes \varepsilon_{-4}$	
			6	$\tau_{\mathrm{sc},2}(5, 1, 3) \otimes \varepsilon_{\pm 8}$	
		8	5	$\tau_{\mathrm{sc},2}(-4, 3, 4)$, $\tau_{\mathrm{sc},2}(-20, 3, 4)$	
			6	$\tau_{\mathrm{sc},2}(-4, 3, 4) \otimes \varepsilon_8$, $\tau_{\mathrm{sc},2}(-20, 3, 4) \otimes \varepsilon_8$	
			8	$\tau_{\mathrm{sc},2}(-4, 6, 4) \otimes \varepsilon_d \quad (d = 1, -4)$	
		24	3	$\tau_{\mathrm{ex},2}$	exceptional supercuspidal
			4	$\tau_{\mathrm{ex},2} \otimes \varepsilon_{-4}$	
			6	$\tau_{\mathrm{ex},2} \otimes \varepsilon_{\pm 8}$	
			7	$\tau_{\mathrm{ex},1} \otimes \varepsilon_d \ (d = 1, -4, \pm 8)$	

TABLE 1. Inertial WD-types for elliptic curves over $\mathbb{Q}_\ell$

$\ell \geq 5$. The remainder of the paper is concerned with additive, potentially good reduction first for $\ell = 3$ (section 5) then $\ell = 2$ (sections 6–7 for the nonexceptional and exceptional cases).

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2. Two-dimensional Weil–Deligne representations

In this section, we quickly recall background on Galois representations of local fields and types. Our main references are Rohrlich [27], Carayol [9], and Bushnell–Henniart [6].

2.1. Notation. Let ℓ be prime and let $F \supseteq \mathbb{Q}_\ell$ be a finite extension with algebraic closure F^{al} and maximal unramified extension $F^{\mathrm{un}} \subset F^{\mathrm{al}}$. Let $\mathcal{O}_F \subset F$ be the valuation ring of F with maximal ideal $\mathfrak{p}$, uniformizer $\varpi \in \mathfrak{p}$, and residue field k of cardinality $q := \#k$. Let $v\colon F^\times \to \mathbb{Z}$ denote the valuation of F normalized with $v(\varpi) = 1$, and let $|\cdot|_v\colon F^\times \to \mathbb{R}^\times_{>0}$ be the associated normalized absolute value. Let $W_F < \mathrm{Gal}(F^{\mathrm{al}} \,|\, F)$ be the Weil group of F and $I_F < W_F$ its inertia subgroup, fitting into the exact sequence

$$1 \to I_F \to W_F \to \mathbb{Z} \to 1. \tag{2.1.1}$$

For $F = \mathbb{Q}_\ell$, for brevity we replace F by ℓ in the subscript, writing e.g. $I_\ell < W_\ell$.

Let W_F^{ab} denote the maximal abelian quotient of W_F and let $\mathrm{Art}_F\colon F^\times \xrightarrow{\sim} W_F^{\mathrm{ab}}$ be the Artin reciprocity map from local class field theory, the isomorphism of topological groups sending $\varpi \in \mathcal{O}_F$ to the class of a geometric Frobenius element $\mathrm{Fr} \in W_F^{\mathrm{ab}}$, characterized by $\mathrm{Fr}(x^q) = x$ for $x \in k$. The map Art_F allows us to identify a character χ of W_F with the character $\chi^{\mathrm{A}} := \chi \circ \mathrm{Art}_F$ of $F^\times$, and conversely. We call a continuous homomorphism $\chi\colon W_F \to \mathbb{C}^\times$ with open kernel a quasicharacter, and if $|\chi(g)| = 1$ for all $g \in W_F$ then we call χ a (unitary) character. The conductor of χ is the ideal $\mathrm{cond}(\chi) := \mathfrak{p}^m$ where $\mathrm{condexp}(\chi) := m \in \mathbb{Z}_{\geq 0}$ is conductor exponent, the smallest nonnegative integer such that the restriction $\chi^{\mathrm{A}}|_{1+\mathfrak{p}^m}$ to $1 + \mathfrak{p}^m \leq \mathcal{O}_K^\times$ is trivial. Let $\omega\colon W_F \to \mathbb{C}^\times$ be the quasicharacter corresponding to the norm quasicharacter $|\cdot|_v$, so that $\omega(g) = q^{-a}$ for $g|_{F^{\mathrm{un}}} = \mathrm{Fr}^a$ with $a \in \mathbb{Z}$.

Definition 2.1.2. A (n-dimensional) Weil–Deligne representation is a pair (ρ, N) such that

- $\rho\colon W_F \to \mathrm{GL}_n(\mathbb{C})$ is a homomorphism with open kernel, and
- $N \in \mathrm{GL}_n(\mathbb{C})$ is nilpotent and satisfies

$$\rho(g) N \rho(g)^{-1} = \omega(g) N \quad \text{for all } g \in W_F. \tag{2.1.3}$$

An isomorphism (or equivalence) of Weil–Deligne representations from (ρ, N) to (ρ', N') is specified by an element $P \in \mathrm{GL}_n(\mathbb{C})$ such that $\rho'(g) = P\rho(g)P^{-1}$ for all $g \in W_F$ and $N' = PNP^{-1}$.

2.2. Classification. Every 2-dimensional Weil–Deligne representation arises up to isomorphism from one of the following three possibilities.

- *Principal series.* Let $\chi_1, \chi_2\colon W_F \to \mathbb{C}^\times$ be quasicharacters such that $\chi_1\chi_2^{-1} \neq \omega^{\pm 1}$. The principal series representation associated to χ_1, χ_2 is $(\mathrm{PS}(\chi_1, \chi_2), 0)$, where

  $$\mathrm{PS}(\chi_1, \chi_2) := \chi_1 \oplus \chi_2.$$

  Its conductor exponent is given by

$$\mathrm{condexp}(\mathrm{PS}(\chi_1, \chi_2)) = \mathrm{condexp}(\chi_1) + \mathrm{condexp}(\chi_2). \tag{2.2.1}$$

- *Special or Steinberg representations.* Let $\chi\colon W_F \to \mathbb{C}^\times$ be a quasicharacter. The **special** or **Steinberg** representation associated to χ is $(\mathrm{St}(\chi), N)$, where

$$\mathrm{St}(\chi) := \chi\omega \oplus \chi \quad \text{and} \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2.2.2}$$

We have

$$\mathrm{condexp}(\mathrm{St}(\chi)) = \begin{cases} 2\,\mathrm{condexp}(\chi), & \text{if } \chi \text{ is ramified;} \\ 1, & \text{otherwise.} \end{cases} \tag{2.2.3}$$

- *Supercuspidal representations.* The Weil–Deligne representations $(\rho, 0)$ where ρ is an *irreducible* 2-dimensional representation of W_F are called **supercuspidal**. Supercuspidal representations are classified by their projective images in $\mathrm{PGL}_2(\mathbb{C})$: see Bushnell–Henniart [6, sections 41 and 42] or Carayol [9, section 12]. We say that ρ is **nonexceptional** (or **imprimitive**) if its projective image is dihedral, otherwise ρ is **exceptional** (or **primitive**) and has projective image A_4 or S_4. (Since W_F is totally disconnected, the projective image A_5 cannot occur.)

2.3. **Supercuspidal representations.** Since they will command significant attention here, we explore supercuspidal representations further.

Nonexceptional supercuspidal representations. Let $K \supseteq F$ be a finite extension and let $\chi\colon W_K \to \mathbb{C}^\times$ be a quasicharacter. We say that $\chi^{\mathrm{A}} := \chi \circ \mathrm{Art}_K$ **factors through the norm map** $\mathrm{Nm}_{K|F}$ if there exists a character $\theta^{\mathrm{A}}\colon F^\times \to \mathbb{C}^\times$ such that

$$\chi^{\mathrm{A}} = \theta^{\mathrm{A}} \circ \mathrm{Nm}_{K|F}\,.$$

Suppose further that $[K : F] = 2$, and let $s \in W_F$ be a lift of the nontrivial element in $\mathrm{Gal}(K\,|\,F)$. Since $W_K \trianglelefteq W_F$ is normal, the s-**conjugate** of χ

$$\begin{aligned} \chi^s\colon W_K &\to \mathbb{C}^\times \\ \chi^s(g) &= \chi(s^{-1}gs), \end{aligned} \tag{2.3.1}$$

is independent of the choice of s, and by local class field theory we have $(\chi^s)^{\mathrm{A}} = \chi^{\mathrm{A}} \circ s$.

Lemma 2.3.2. *We have $\chi = \chi^s$ if and only if χ^{A} factors through the norm map.*

Proof. We claim χ^{A} factors through the norm map if and only if $\ker \mathrm{Nm}_{K|F}(K^\times) \subseteq \ker \chi^{\mathrm{A}}$: the direction $(\Rightarrow)$ is clear, and to show $(\Leftarrow)$, the assignment $\theta^{\mathrm{A}}(x) = \chi^{\mathrm{A}}(y)$ if $\mathrm{Nm}_{K|F}(y) = x$ is well-defined, and we may then extend from $\mathrm{Nm}_{K|F}(K^\times)$ to $F^\times$. By Hilbert's Theorem 90, $\ker \mathrm{Nm}_{K|F}(K^\times) = \{x/s(x) : x \in K^\times\}$, and $\chi^{\mathrm{A}}(x/s(x)) = 1$ if and only if $\chi^{\mathrm{A}}(x) = \chi^{\mathrm{A}}(s(x))$ for $x \in K^\times$. Thus χ^{A} factors through the norm map if and only if $\chi^{\mathrm{A}}(x) = \chi^{\mathrm{A}}(s(x)) = (\chi^s)^{\mathrm{A}}(x)$ for all $x \in K^\times$ if and only if $\chi = \chi^s$. □

Suppose that $\chi \neq \chi^s$. Then the **nonexceptional supercuspidal** representation attached to χ is $(\mathrm{Ind}_{W_K}^{W_F} \chi, 0)$, the induction of χ from W_K to W_F. When no confusion can result, we write simply $\mathrm{Ind}\,\chi$. The condition $\chi \neq \chi^s$ is necessary to ensure that $\mathrm{Ind}\,\chi$ is irreducible. If ψ_K is the quadratic character of W_F corresponding to K, then

$$\mathrm{condexp}(\mathrm{Ind}\,\chi) = \begin{cases} 2\,\mathrm{condexp}(\chi), & \text{if } K\,|\,F \text{ is unramified;} \\ \mathrm{condexp}(\chi) + \mathrm{condexp}(\psi_K), & \text{otherwise.} \end{cases} \tag{2.3.3}$$

When $\det(\operatorname{Ind}\chi) = \omega$, then χ satisfies

$$(\chi^{\mathrm{A}}|_{F^\times}) \cdot \omega^{\mathrm{A}} = |\cdot|_v \tag{2.3.4}$$

as quasicharacters of $F^\times$.

Exceptional supercuspidal representations. Exceptional representations only exist for residual characteristic $\ell = 2$, so suppose that $\ell = 2$. Let $L \supseteq F$ be a (tamely) ramified cubic extension, and let $M \supseteq L$ be a ramified quadratic extension (ramified is necessary, see [6, §42.1, Proposition, p. 257, part (1)]). Let χ be a character of W_M such that χ^{A} does not factor through the norm map $\mathrm{Nm}_{M|L}$. Given the data (L, M, χ), by Bushnell–Henniart [6, p. 261] there is a exceptional supercuspidal Weil–Deligne representation $(\rho, 0)$ such that

$$\rho|_{W_L} = \operatorname{Ind}_{W_M}^{W_L} \chi. \tag{2.3.5}$$

Conversely, every exceptional supercuspidal representation is uniquely determined by such a triple (L, M, χ), up to equivalence [9, Lemme 12.1.3].

2.4. **Inertial types.** An inertial Weil–Deligne (or WD-)type is an equivalence class of Weil–Deligne representations (ρ, N) under the equivalence relation $(\rho, N) \sim (\rho', N')$ if and only if there exists $P \in \mathrm{GL}_2(\mathbb{C})$ such that $\rho'(g) = P\rho(g)P^{-1}$ and $N' = PNP^{-1}$ for all $g \in I_F$. (The content is in the restriction to $g \in I_F$; we might think of this as being an equivalence of Weil–Deligne representations over F^{un}.) Such an equivalence class is determined by the pair (τ, N) where $\tau = \rho|_{I_F}$ is the (common) restriction to I_F for a WD-type, with the evident notion of equivalence, so this definition agrees with the one given in the introduction. Except for the special (Steinberg) representations we have $N = 0$, so (except in the short section 4.1) we drop N from the notation and write simply τ.

We record the following classification of all inertial WD-types.

Proposition 2.4.1. *Let $\tau\colon I_F \to \mathrm{GL}_2(\mathbb{C})$ be an inertial WD-type. Then exactly one of the following holds:*

(i) *τ is the restriction of a principal series, i.e., there exist $\chi_1, \chi_2\colon W_F \to \mathbb{C}^\times$ such that*
$$\tau = \mathrm{PS}(\chi_1, \chi_2)|_{I_F} = \chi_1|_{I_F} \oplus \chi_2|_{I_F};$$

(ii) *$\tau = \mathrm{St}(\chi)|_{I_F}$ is the restriction of a special series for χ a character of W_F;*

(iii) *There exists a character $\chi\colon W_K \to \mathbb{C}^\times$, where $K \supseteq F$ is the unramified quadratic extension, such that $\chi \neq \chi^s$ and*
$$\tau = (\operatorname{Ind}_{W_K}^{W_F} \chi)|_{I_F} = \chi|_{I_F} \oplus \chi^s|_{I_F};$$

(iv) *There exist a ramified quadratic extension $K \supseteq F$, and a character $\chi\colon W_K \to \mathbb{C}^\times$ such that $\chi|_{I_K} \neq \chi^s|_{I_K}$ and*
$$\tau = \operatorname{Ind}_{I_K}^{I_F}(\chi|_{I_F}); \text{ or}$$

(v) *τ is the restriction of an exceptional supercuspidal Weil–Deligne representation.*

Proof. For (i)–(iv), see Breuil–Mézard [5, Lemme 2.1.1.2, Théorème 2.1.1.4] and for (v) see Bushnell–Henniart [6, §41 and §42]. □

According to Proposition 2.4.1, we may say that an inertial type $[\rho, N]$ is principal series, special, or (nonexceptional or exceptional) supercuspidal according as (ρ, N).

2.5. **Notation.** We conclude this section with the notation we will use throughout, in one place for convenience.

- We write $\varepsilon_d \colon I_{\mathbb{Q}_\ell} \to \mathbb{C}^\times$ for the quadratic character associated to the (ramified) quadratic extension $\mathbb{Q}_\ell(\sqrt{d})$ of discriminant $d \in \mathbb{Z}_\ell$ (well-defined up to $\mathbb{Z}_\ell^{\times 2}$).
- We write $\tau_{\mathrm{St},\ell}$ to denote the special (Steinberg) type (2.2.2), with nonzero nilpotent monodromy operator N; in all other cases, $N = 0$.
- To identify the nonspecial, nonexceptional types, we use the notation

$$\tau_*(d, f, r)_j := \left(\mathrm{Ind}_{W_{\mathbb{Q}_\ell(\sqrt{d})}}^{W_{\mathbb{Q}_\ell}} \chi_{(d,f,r)}\right)|_{I_\ell} \tag{2.5.1}$$

  where:
  - $* \in \{\mathrm{ps}, \mathrm{sc}\}$ is either principal series or (nonexceptional) supercuspidal;
  - d is the discriminant of $K := \mathbb{Q}_\ell(\sqrt{d})$, with $[K : \mathbb{Q}_\ell] \le 2$;
  - for $\ell \neq 2$, let $u \in \mathbb{Z}_\ell^\times \smallsetminus \mathbb{Z}_\ell^{\times 2}$ be a nonsquare;
  - $\chi_{(d,f,r)} \colon W_K \to \mathbb{C}^\times$ is a character;
  - f is the conductor exponent of the character χ (as a power of the maximal ideal in the ring of integers of K);
  - r is the order of the character χ on the inertia subgroup $I_K \subset W_K$; and
  - j is an additional label (only needed for $\ell = 3$, see Table 4).
- For $\ell = 2$, two exceptional (octahedral) representations $\tau_{\mathrm{ex},i}$ for $i = 1, 2$ are explicitly given (see section 7.2).

3. Background on elliptic curves

In this section we organize some facts about elliptic curves and provide a few preliminary results on their inertial types. Throughout this section, let E be an elliptic curve over F with conductor N_E.

3.1. **Inertial types.** There is a Weil–Deligne representation (ρ_E, N) attached to E which is obtained as follows (for complete details we refer to [27, §4 and §13–15]). We start with the representation $\rho_{E,p} : \mathrm{Gal}(F^{\mathrm{al}} \,|\, F) \to \mathrm{GL}_2(\mathbb{Q}_p)$ defined by the action of $\mathrm{Gal}(F^{\mathrm{al}} \,|\, F)$ on the étale cohomology group $H^1_{\mathrm{et}}(E \times_F F^{\mathrm{al}}, \mathbb{Q}_p)$ for some prime $p \neq \ell$; this is equivalent to working with the p-adic Tate module but with the contragredient representation, i.e., into the dual of the p-adic Tate module.

Next, we consider two cases: either E has potentially good reduction, hence $\rho_{E,p}(I_F)$ has finite order, or E has potentially multiplicative reduction and so $\rho_{E,p}(I_F)$ is infinite. In the first case, we take $N = 0$ and ρ_E is obtained by extension of scalars of the restriction $\rho_{E,p}|_{W_F}$ via an embedding $\iota \colon \mathbb{Q}_p \hookrightarrow \mathbb{C}$; the $\mathbb{C}$-equivalence class is well-defined, independent of choices. When E has potentially multiplicative reduction, then $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\rho_E = \mathrm{St}(\chi)$ where χ is the quadratic character of W_F such that $E \otimes \chi$ has split multiplicative reduction—note this is the only type arising from a representation with nontrivial nilpotent endomorphism. In either case, we define the inertial WD-type τ_E of E to be the equivalence class $\tau_E = [\rho_E, N]$ as defined in section 2.4. Finally, we note that the conductors of $\rho_{E,p}$ and τ_E are both equal to N_E (see e.g. [27, §18] and [13, Remark 2.14]).

Example 3.1.1. For a quadratic extension $\mathbb{Q}_\ell(\sqrt{d}) \supseteq \mathbb{Q}_\ell$ of discriminant d and character χ_d, let E_d be the quadratic twist of $E/\mathbb{Q}_\ell$ by d. The inertial type τ_d of E_d satisfies $\tau_d \simeq \tau_E \otimes \varepsilon_d$, where $\varepsilon_d = \chi_d|_{I_\ell}$ is the restriction of the twisting character to inertia. Note the nilpotent operator remains unchanged as $\rho_{E,p}(I_\ell)$ is finite if and $\rho_{E_d,p}(I_\ell) = (\rho_{E,p} \otimes \chi_d)(I_\ell)$ is finite.

The following summarizes the possibilites for N_E in the case $F = \mathbb{Q}_\ell$.

Lemma 3.1.2. *Let $E/\mathbb{Q}_\ell$ be an elliptic curve. We have*

$$0 \leq \mathrm{ord}_\ell(N_E) \leq \begin{cases} 2, & \text{if } \ell \geq 5; \\ 5, & \text{if } \ell = 3; \\ 8, & \text{if } \ell = 2. \end{cases}$$

Moreover, if E has additive reduction then $\ell^2 \mid N_E$.

Proof. See e.g. Silverman [30, Ch. IV, Theorem 10.4]. □

Remark 3.1.3. Elliptic curves defined over ramified extensions $F/\mathbb{Q}_2$ or $F/\mathbb{Q}_3$ can have conductors whose valuations are higher than those given by Lemma 3.1.2.

3.2. **Potentially good reduction.** We set up some preparatory facts in this section. Suppose throughout that E/F has potentially good reduction and inertial type τ. Thus $N = 0$.

Let $m \in \mathbb{Z}_{\geq 3}$ be coprime to ℓ, and let $L := F^{\mathrm{un}}(E[m])$ where F^{un} is the maximal unramified extension of F. The extension L is independent of m (see Serre-Tate [29, §2, Corollary 3]) and it has two other equivalent descriptions:

- L is the minimal extension of F^{un} where E achieves good reduction; and
- L is the fixed field of $\ker \tau$.

We call L the inertial field of E. Write $\Phi := \mathrm{Gal}(L \,|\, F^{\mathrm{un}})$ and define the semistability defect of E to be $e = e(E/F) := \#\Phi$. The following describes the possibilities for Φ.

Lemma 3.2.1. *Exactly one of the following possibilities hold.*

(i) *Φ is cyclic of order $2, 3, 4, 6$.*
(ii) *$\ell = 3$ and $\Phi \simeq \mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$ is of order 12;*
(iii) *$\ell = 2$ and $\Phi \simeq Q_8$ is isomorphic to a quaternion group of order 8; or*
(iv) *$\ell = 2$ and $\Phi \simeq \mathrm{SL}_2(\mathbb{F}_3)$ is of order 24.*

Proof. See Kraus [21, pp. 354–357]. □

The next lemma already determines the inertial type of $E/\mathbb{Q}_\ell$ in the simplest case.

Lemma 3.2.2. *Suppose $e(E/\mathbb{Q}_\ell) = 2$. Then τ is principal series, and the following statements hold.*

(a) *If $\ell \geq 3$, then $N_E = \ell^2$ and $\tau \simeq \mathrm{PS}(\chi_\ell, \chi_\ell)|_{I_\ell} \simeq \varepsilon_\ell$.*
(b) *If $\ell = 2$, then $N_E = 2^4, 2^6$, and*

$$\tau \simeq \begin{cases} \varepsilon_{-4}, & \text{if } N_E = 2^4; \\ \varepsilon_{\pm 8}, & \text{if } N_E = 2^6. \end{cases}$$

Proof. Since $e(E/\mathbb{Q}_\ell) = 2$, there exists a ramified quadratic extension $\mathbb{Q}_\ell(\sqrt{d}) \supseteq \mathbb{Q}_\ell$ such that the quadratic twist E_d has good reduction (see e.g. Freitas–Kraus [19, Lemmas 3–4]) and therefore its inertial type τ_d is trivial. Thus

$$\tau \simeq \tau_{E_d} \otimes \varepsilon_d \simeq \varepsilon_d \simeq \mathrm{PS}(\chi_d, \chi_d)|_{I_\ell}$$

is principal series. We have $d = \ell$ if $\ell \geq 3$; if $\ell = 2$, we have $d = -4, \pm 8$. The claim on the conductor follows from $\operatorname{ord}_\ell(N_E) = \operatorname{condexp}(\tau) = 2\operatorname{cond}(\chi_d)$ by (2.2.1). ☐

We conclude this short section with a preliminary step to determine all the exceptional supercuspidal types arising from elliptic curves over $\mathbb{Q}_2$; these will be given explicitly in Section 7.

Lemma 3.2.3. *Suppose $F = \mathbb{Q}_2$ and E has potentially good reduction. Then τ is exceptional supercuspidal if and only if $e = 24$.*

Proof. Suppose τ is exceptional. From the group structure of the image of the projective representation obtained by postcomposing with $\mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$, by Bushnell–Henniart [6, section 42.3] it follows that $e \geq 12$, so $e = 24$ by Lemma 3.2.1.

Conversely, suppose $e = 24$, and let $\rho_{E,3}\colon W_2 \to \mathrm{GL}_2(\mathbb{Q}_3)$ and $\overline{\rho}_3\colon W_2 \to \mathrm{GL}_2(\mathbb{F}_3)$ respectively be the 3-adic and mod 3 Galois representations associated to E, restricted to W_2.

By Dokchitser–Dokchitser [17, Lemma 1], there is an unramified twist of $\rho_{E,3}$ factoring through the Galois group of $K := \mathbb{Q}_2(E[3])$ (over $\mathbb{Q}_2$), so the images of the projective representations $\mathrm{P}\rho_{E,3}\colon W_2 \to \mathrm{PGL}_2(\mathbb{Q}_3)$ and $\mathrm{P}\overline{\rho}_3\colon W_2 \to \mathrm{PGL}_2(\mathbb{F}_3)$ are isomorphic as abstract groups. By hypothesis and Lemma 3.2.1, τ is irreducible with image isomorphic to $\Phi \simeq \mathrm{SL}_2(\mathbb{F}_3)$, so $\overline{\rho}_3(W_2) = \mathrm{GL}_2(\mathbb{F}_3)$ is surjective [17, Table 1]. Thus $\mathrm{P}\rho_{E,3}$ has image isomorphic to $\mathrm{PGL}_2(\mathbb{F}_3) \simeq S_4$, so $\rho_{E,3} \otimes \mathbb{C}$ and hence τ are exceptional supercuspidal. ☐

4. Inertial types: uniform cases

Beginning in this section and continuing through the rest of the paper, we seek to describe explicitly the inertial types arising from elliptic curves over $\mathbb{Q}_\ell$. In this section, we treat two cases where the answer is close to uniform in ℓ: elliptic curves with potentially multiplicative reduction and the case where $\ell \geq 5$. Throughout, we use the notation collected in section 2.5.

4.1. **Potentially multiplicative reduction and special types.** We begin with a general result on inertial types for elliptic curves with potentially multiplicative reduction. These are the only types with a nonzero nilpotent operator, more precisely $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

Proposition 4.1.1. *Let E be an elliptic curve over $\mathbb{Q}_\ell$ with potentially multiplicative reduction, conductor N_E, and inertial type τ. Then the following statements hold.*

(a) *If E has multiplicative reduction, then $N_E = \ell$ and $\tau \simeq \tau_{\mathrm{St},\ell}$ is special.*

(b) *Suppose E has additive (but potentially multiplicative) reduction. Then $\ell^2 \mid N_E$, and τ is special. Moreover:*
   (i) *If $\ell \geq 3$, then $N_E = \ell^2$ and $\tau \simeq \tau_{\mathrm{St},\ell} \otimes \varepsilon_\ell$.*
   (ii) *If $\ell = 2$, then $N_E = \ell^4, \ell^6$, and*

$$\tau \simeq \begin{cases} \tau_{\mathrm{St},2} \otimes \varepsilon_{-4}, & \text{if } N_E = \ell^4; \\ \tau_{\mathrm{St},2} \otimes \varepsilon_{\pm 8} \text{ or } \tau_{\mathrm{St},2} \otimes \varepsilon_{\pm 8} & \text{if } N_E = \ell^6. \end{cases}$$

Proof. We recall from section 3.1 that $\rho_E = \mathrm{St}(\chi)$ and $\tau = \mathrm{St}(\chi)|_{I_\ell}$ for some quadratic character χ of W_ℓ.

In part (a), we have $N_E = \operatorname{cond}(\tau) = \ell$ and by the conductor formula (2.2.3) it follows that χ is unramified; in this case,

$$\tau = (\chi \otimes \mathrm{St}(1))|_{I_\ell} = \chi|_{I_\ell} \otimes \mathrm{St}(1)|_{I_\ell} = \mathrm{St}(1)|_{I_\ell} = \tau_{\mathrm{St},\ell}. \tag{4.1.2}$$

We turn to part (b). We have $\ell^2 \mid N_E$, and formula (2.2.3) gives $N_E = \mathrm{cond}(\tau) = \ell^{2m}$, so χ is ramified with $\mathrm{cond}(\chi) = \ell^m$. If $\ell \geq 3$, then any quadratic character $\chi\colon W_\ell \to \mathbb{C}^\times$ has conductor ℓ and satisfies $\chi|_{I_\ell} = \varepsilon_\ell$. Thus $N_E = \ell^2$, and $\tau \simeq \tau_{\mathrm{St},\ell} \otimes \varepsilon_\ell$, proving (i). Otherwise, we have $\ell = 2$, and we conclude similarly, as in Lemma 3.2.2: we have $\chi|_{I_\ell} = \varepsilon_{-4}, \varepsilon_{\pm 8}$ of conductors $2^2, 2^3$, proving (ii). □

4.2. **Inertial types for $\ell \geq 5$.** The preceding section treated all cases of potentially multiplicative reduction, so for the rest of this paper we turn to the case of potentially good reduction. In particular, $N = 0$. Here we treat the case $\ell \geq 5$ where the results are uniform.

Proposition 4.2.1. *Let $\ell \geq 5$. Let $E/\mathbb{Q}_\ell$ be an elliptic curve with additive potentially good reduction, semistability defect $e \geq 3$, and inertial type τ. Then the following statements hold.*

(a) *If $e \mid (\ell - 1)$, then $\tau \simeq \tau_{\mathrm{ps},\ell}(1, 1, e)$ is principal series.*
(b) *If $e \mid (\ell + 1)$, then $\tau \simeq \tau_{\mathrm{sc},\ell}(u, 2, e)$ is supercuspidal.*

Proof. Lemma 3.1.2 implies that τ has conductor ℓ^2, and Lemma 3.2.1 shows that the image of τ is cyclic of order $e = 3, 4, 6$. From the classification in Proposition 2.4.1, τ is reducible with finite image, hence it is either principal series or nonexceptional supercuspidal induced from the unramified quadratic extension $F = \mathbb{Q}_{\ell^2}$ of $\mathbb{Q}_\ell$.

Suppose that τ is principal series. Then, $\tau = \chi|_{I_\ell} \oplus \chi^{-1}|_{I_\ell}$, where χ is a character of W_ℓ of conductor ℓ and order e. To ease notation we write χ also for χ^A. Thus, $\chi|_{I_\ell}$ factors through $(\mathbb{Z}_\ell/\ell\mathbb{Z}_\ell)^\times \simeq \mathbb{F}_\ell^\times$ a cyclic group of order $\ell - 1$, so $e \mid (\ell - 1)$. Let $u \in \mathbb{Z}_\ell^\times$ and χ_e be as in section 2.5. So the reduction of u generates $\mathbb{F}_\ell^\times$. We have $\chi(u) = \zeta_e^c = \exp(2\pi i c/e)$ with $\gcd(c, e) = 1$. Since $e = 3, 4, 6$, we must have $c \equiv \pm 1 \pmod{e}$, so $\chi|_{I_\ell} = \chi_e^{\pm 1}$ and either choice gives $\tau \simeq \tau_{\mathrm{ps},\ell}(1, 1, e)$.

To finish, suppose τ is supercuspidal. Then, $\tau = \chi|_{I_F} \oplus \chi^{-1}|_{I_F}$, where χ is a character of W_F of order e. Since τ has conductor $\ell^2\mathcal{O}_F$, it follows that χ viewed as a character of $F^\times$ has conductor $\ell\mathcal{O}_F$ and satisfies $\chi|_{\mathbb{Z}_\ell^\times} = 1$ by (2.3.4). Now $\chi|_{I_F}$ factors through $(\mathbb{Z}_{\ell^2}/\ell\mathbb{Z}_{\ell^2})^\times \simeq \mathbb{F}_{\ell^2}^\times$ so $e \mid (\ell^2 - 1) = (\ell + 1)(\ell - 1)$. Let $u \in \mathbb{Z}_{\ell^2}^\times$ be as in section 2.5, so its reduction generates $\mathbb{F}_{\ell^2}^\times$. Then $u^{\ell+1}$ generates $(\mathbb{Z}_\ell/\ell\mathbb{Z}_\ell)^\times \leq (\mathbb{Z}_{\ell^2}/\ell\mathbb{Z}_{\ell^2})^\times$. The condition $\chi|_{\mathbb{Z}_\ell^\times} = 1$ implies that $e \mid (\ell + 1)$. We again have $\chi(u) = \zeta_e^c$ and as in the previous paragraph we must have $\chi|_{I_F} = \chi_e^{\pm 1}$ and either choice gives $\tau \simeq \tau_{\mathrm{sc},\ell}(u, 2, e)$. □

5. Inertial types for $\ell = 3$

In this section, we treat the case $\ell = 3$; see Breuil–Conrad–Diamond–Taylor [4] for the application of these types to the modularity of elliptic curves.

5.1. **Setup.** Throughout this section, we let $K = \mathbb{Q}_3$ or $K = \mathbb{Q}_3(\sqrt{d})$ where $d = -1, \pm 3$. Let $\mathcal{O}_K$ be the valuation ring of K and $\mathfrak{p}$ its maximal ideal. When K is quadratic, let $\chi_d\colon W_3 \to \mathbb{C}^\times$ be the quadratic character associated to K and let $s \in W_3$ be a lift of the nontrivial element of $\mathrm{Gal}(K \mid \mathbb{Q}_3)$. Recall that $\varepsilon_3 = \chi_{\pm 3}|_{I_3}$ is the unique nontrivial quadratic character of I_3; we have $\mathrm{cond}(\varepsilon_3) = 3$.

We begin with some explicit class field theoretic results on structures of finite quotients of $\mathcal{O}_K^\times$.

Lemma 5.1.1. *Let $K/\mathbb{Q}_3$ be one of the extensions above, $\mathfrak{f} = \mathfrak{p}^k$ and $f = \mathbb{Z}_3 \cap \mathfrak{f}$, with $k \geq 1$ integer. Table 2 gives the structure and explicit generators for the group $(\mathcal{O}_K/\mathfrak{f})^\times/U$, where $U = \{1\}$ if $K = \mathbb{Q}_3$ and $U = \mathrm{Nm}_{K|\mathbb{Q}_3}((\mathcal{O}_K/\mathfrak{f})^\times)$, $\mathrm{Nm}_{K|\mathbb{Q}_3} : (\mathcal{O}_K/\mathfrak{f})^\times \to (\mathbb{Z}_3/f)^\times$, otherwise.*

K	$\mathfrak{f}$	$(\mathcal{O}_K/\mathfrak{f})^\times/U$	Group structure
$\mathbb{Q}_3$	$3^k(k \geq 1)$	$\langle -4 \rangle$	$\mathbb{Z}/(2 \cdot 3^{k-1})$
$\mathbb{Q}_3(\sqrt{-1})$	$3^k(k \geq 1)$	$\langle \sqrt{-1}+2 \rangle$	$\mathbb{Z}/(4 \cdot 3^{k-1})$
$\mathbb{Q}_3(\sqrt{3})$	$\mathfrak{p}^k(k \geq 1)$	$\langle \sqrt{3}-1 \rangle$	$\mathbb{Z}/(2 \cdot 3^{\lfloor \frac{k}{2} \rfloor})$
$\mathbb{Q}_3(\sqrt{-3})$	$\mathfrak{p}^k(k = 1,2,3)$	$\langle -w+4 \rangle$	$\mathbb{Z}/(2 \cdot 3^{\lfloor \frac{k}{2} \rfloor})$
	$\mathfrak{p}^k(k \geq 4)$	$\langle w-1 \rangle \times \langle -w+4 \rangle$	$\mathbb{Z}/3 \times \mathbb{Z}/(2 \cdot 3^{\lfloor \frac{k-2}{2} \rfloor})$

TABLE 2. The group structure of the group $(\mathcal{O}_K/\mathfrak{f})^\times/U$ $(w = \frac{1+\sqrt{-3}}{2})$.

Proof. The proof is by induction on k. □

Lemma 5.1.2. *Let $K/\mathbb{Q}_3$ be one of the quadratic extensions above, and $\mathfrak{f} = \mathfrak{p}^k$, with $k \geq 1$ integer. Let $U = \mathrm{Nm}_{K|\mathbb{Q}_3}((\mathcal{O}_K/\mathfrak{f})^\times)$ and $f = \mathbb{Z}_3 \cap \mathfrak{f}$. Then, we have*

(i) $(\mathbb{Z}_3/f)^\times = U$ if $K = \mathbb{Q}_3(\sqrt{-1})$;
(ii) $(\mathbb{Z}_3/f)^\times/U = \langle -1 \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ if $K = \mathbb{Q}_3(\sqrt{\pm 3})$.

Proof. The proof uses induction on k the same way Lemma 5.1.1 does. □

Keeping the above notations, let $H := \ker(\mathrm{Nm}_{K|\mathbb{Q}_3})$, and recall that $U = \mathrm{Im}(\mathrm{Nm}_{K|\mathbb{Q}_3})$. Both H and U are subgroups of $(\mathcal{O}_K/\mathfrak{f})^\times$.

Lemma 5.1.3. *Let $K/\mathbb{Q}_3$ be one of the quadratic extensions above, and $\mathfrak{f} = \mathfrak{p}^k$, with $k \geq 1$ integer. Let $U = \mathrm{Nm}_{K|\mathbb{Q}_3}((\mathcal{O}_K/\mathfrak{f})^\times)$ and $H := \ker(\mathrm{Nm}_{K|\mathbb{Q}_3})$, and recall the exact sequence*

$$1 \to U \to (\mathcal{O}_K/\mathfrak{f})^\times \to (\mathcal{O}_K/\mathfrak{f})^\times/U \to 1.$$

Then $H \simeq (\mathcal{O}_K/\mathfrak{f})^\times/U$, and the group structures of U and $H \cap U$ are given in Table 3.

K	$\mathfrak{f}$	U	Group structure of U	$H \cap U$
$\mathbb{Q}_3(\sqrt{-1})$	$3^k(k \geq 1)$	$\langle 4 \rangle$	$\mathbb{Z}/(3^{k-1})$	$\langle -1 \rangle$
$\mathbb{Q}_3(\sqrt{3})$	$\mathfrak{p}^k(k \geq 1)$	$\langle 4 \rangle$	$\mathbb{Z}/(3^{\lfloor \frac{k-1}{2} \rfloor})$	$\{1\}$
$\mathbb{Q}_3(\sqrt{-3})$	$\mathfrak{p}^k(k = 1,2,3)$	$\langle 4 \rangle$	$\mathbb{Z}/(3^{\lfloor \frac{k}{2} \rfloor})$	$\{1\}$
	$\mathfrak{p}^k(k \geq 4)$	$\langle 4 \rangle$	$\mathbb{Z}/(3^{\lfloor \frac{k-1}{2} \rfloor})$	$\{1\}$

TABLE 3. The group structures of U and $H \cap U$.

Proof. The proof is an induction which combines Lemmas 5.1.1 and 5.1.2. □

Using Lemma 5.1.3, we identify $H = \ker(\mathrm{Nm}_{K|\mathbb{Q}_3})$ with $(\mathcal{O}_K/\mathfrak{f})^\times/U$.

Corollary 5.1.4. *Let $K/\mathbb{Q}_3$ be one of the quadratic extensions above, and $\mathfrak{f} = \mathfrak{p}^k$, with $k \geq 1$ integer. Let $U = \mathrm{Nm}_{K|\mathbb{Q}_3}((\mathcal{O}_K/\mathfrak{f})^\times)$, and $\chi : (\mathcal{O}_K/\mathfrak{f})^\times \to \mathbb{C}^\times$ a character. Then, we have*

(i) For $K = \mathbb{Q}_3(\sqrt{-1})$, $\chi|_{\mathbb{Z}_3^\times} = \epsilon_K \iff \chi|_U = 1 \iff \chi(4) = \chi(-1) = 1$.
(ii) For $K = \mathbb{Q}_3(\sqrt{\pm 3})$, $$\chi|_{\mathbb{Z}_3^\times} = \epsilon_K \iff \chi|_U = 1 \text{ and } \chi(-1) = -1 \iff \chi(4) = 1 \text{ and } \chi(-1) = -1.$$

Proof. This follows from Lemmas 5.1.2 and 5.1.3. ☐

Corollary 5.1.5. *Let $K/\mathbb{Q}_3$ be one of the ramified quadratic extensions above, and $\mathfrak{f} = \mathfrak{p}^k$, with $k \geq 1$ integer. Let $U = \mathrm{Nm}_{K|\mathbb{Q}_3}((\mathcal{O}_K/\mathfrak{f})^\times)$, and $\chi : (\mathcal{O}_K/\mathfrak{f})^\times \to \mathbb{C}^\times$ a character. Then, χ factors through the norm map if and only if $\chi|_H$ is trivial.*

Proof. This is a direct consequence of Lemmas 5.1.1, 5.1.2 and 5.1.3. ☐

We recall that $\chi_{(d,f,r)}$ denotes a character of W_K where K has discriminant d, whose order on I_K is r and conductor exponent is f. When $\chi_{(d,f,r)}|_U = \epsilon_K$, it is enough to give its values on the generators of $(\mathcal{O}_K/\mathfrak{f})^\times/U$ by Corollaries 5.1.4 and 5.1.5. We fix $\zeta_6 = -\zeta_3^2$ so that $\zeta_6^2 = \zeta_3$. In Table 4, we list the types for $\mathbb{Q}_3$, with generators of $(\mathcal{O}_K/\mathfrak{f})^\times/U$ being as in Lemma 5.1.1 (see also Corollaries 5.1.5 and 5.1.4).

K	$\mathfrak{f}$	$\chi_{(d,f,r)}$ on generators	τ	condexp(τ)
$\mathbb{Q}_3$	3^2	ζ_3	$\tau_{\mathrm{ps},3}(1,2,3)$	4
$\mathbb{Q}_3(\sqrt{-1})$	3	ζ_4	$\tau_{\mathrm{sc},3}(-1,1,4)$	2
	3^2	ζ_3	$\tau_{\mathrm{sc},3}(-1,2,3)$	4
$\mathbb{Q}_3(\sqrt{3})$	$(\sqrt{3})^2$	$-\zeta_3^2$	$\tau_{\mathrm{sc},3}(3,2,6)$	3
$\mathbb{Q}_3(\sqrt{-3})$	$(\sqrt{-3})^2$	$-\zeta_3^2$	$\tau_{\mathrm{sc},3}(-3,2,6)$	3
	$(\sqrt{-3})^4$	$\zeta_3,\ -\zeta_3^{2j}$	$\tau_{\mathrm{sc},3}(-3,4,6)_j$ $(j=0,1,2)$	5

TABLE 4. Types for $\mathbb{Q}_3$

5.2. **Result.** The hard work now begins, with the following proposition.

Proposition 5.2.1. *Let E be an elliptic curve over $\mathbb{Q}_3$ of conductor N_E and inertial type τ. Suppose that E has additive, potentially good reduction and semistability defect $e \geq 3$. Then τ is (nonexceptional) supercuspidal and is given by one of the following cases:*

(a) *If $N_E = 3^2$, then $e = 4$ and $\tau \simeq \tau_{\mathrm{sc},3}(-1,1,4)$.*
(b) *If $N_E = 3^3$, then $e = 12$ and $\tau = \tau_{\mathrm{sc},3}(3,2,6)$ or $\tau = \tau_{\mathrm{sc},3}(-3,2,6)$.*
(c) *If $N_E = 3^4$, then $e = 3, 6$, and:*
  (i) *If $e = 3$, then $\tau \simeq \tau_{\mathrm{ps},3}(1,2,3), \tau_{\mathrm{sc},3}(-1,2,3)$;*
  (ii) *If $e = 6$, then $\tau \simeq \tau_{\mathrm{ps},3}(1,2,3) \otimes \varepsilon_3, \tau_{\mathrm{sc},3}(-1,2,3) \otimes \varepsilon_3$.*
(d) *If $N_E = 3^5$, then $e = 12$ and $\tau = \tau_{\mathrm{sc},3}(-3,4,6)_j$ with $j = 0,1,2$.*

Proof (a). Suppose $N_E = 3^2$. Since the conductor exponent is 2, we are in the case of tame reduction. Hence, $e = 4$ and Φ is cyclic by Lemma 3.2.1. Assume for purposes of contradiction that τ is a principal series; then $\tau \simeq \chi|_{I_3} \oplus \chi^{-1}|_{I_3}$, where χ is a character of W_3 of conductor 3 and order 4, which is impossible. So, τ must be nonexceptional supercuspidal. Noting that $e = 4 \mid (3+1)$, we conclude that $\tau \simeq \tau_{\mathrm{sc},3}(-1,1,4)$ by a similar argument as in the proof of Proposition 4.2.1(b). ☐

Proof (b) *and* (d). Suppose $N_E = 3^m$ with $m = 3$ or $m = 5$. Since $m = \mathrm{condexp}(\tau) > 1$ is odd, τ is obtained by induction of a character χ from a ramified quadratic extension $K \mid \mathbb{Q}_3$ ([7, Chap. IV, §15]). By Proposition 2.4.1 (iv), we conclude that τ is irreducible; hence, $e = 12$ by Lemma 3.2.1. Let $d = \pm 3$ be the discriminant of K. Then, by the conductor formulas, $m = \mathrm{condexp}(\chi) + v_3(d) = \mathrm{condexp}(\chi) + 1$. So, we conclude that $\mathrm{condexp}(\chi) = 2, 4$ according as $m = 3, 5$.

Suppose $m = 3$. By Lemma 5.1.1, we have

$$(\mathcal{O}_K/\mathfrak{p}^2)^\times/U = \langle u \rangle \simeq \mathbb{Z}/6\mathbb{Z},$$

where $u = \sqrt{3} - 1, -w + 4$ for $d = 3, -3$, respectively.

Since χ is primitive, we must have $\chi(u^2) = \zeta_3^{\pm 1}$, hence $\chi(u) = \pm\zeta_3^j$, with $j = 1, 2$. By Corollary 5.1.5, χ does not factor through the norm. Furthermore, since $\chi|_{\mathbb{Z}_3^\times} = \epsilon_K$, Corollaries 5.1.4 implies that $\chi(u) = -\zeta_3^j$, with $j = 1, 2$. This gives two conjugate characters of I_K. Thus, we can take $\chi|_{I_K} = \chi_{(d,2,6)}$, giving $\tau \simeq \tau_{\mathrm{sc},3}(d, 2, 6)$.

Next, suppose that $m = 5$ and $d = 3$. From Lemma 5.1.1 and Corollary 5.1.5, $\chi|_{I_K}$ must induce a non-trivial character on

$$(\mathcal{O}_K/\mathfrak{p}^4)^\times/U = \langle u \rangle \simeq \mathbb{Z}/18\mathbb{Z}$$

where $u = \sqrt{3} - 1$. Moreover, the order of $\chi(u)$ is not a divisor of 6, otherwise χ would have conductor exponent ≤ 3. $\chi|_{I_K}$ has order 9 or 18 and so $9 \mid e = 12$, a contradiction.

Finally, suppose $m = 5$ and $d = -3$. From Lemma 5.1.1 and Corollary 5.1.5, $\chi|_{I_K}$ must induce a non-trivial character on

$$(\mathcal{O}_K/\mathfrak{p}^4)^\times/U = \langle u_1 \rangle \times \langle u_2 \rangle \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$$

where $u_1 = w - 1$ and $u_2 = -w + 4$. The primitivity of χ implies that $\chi(u_1) = \zeta_3^{\pm 1}$. By Corollaries 5.1.4 and 5.1.5, the condition $\chi|_{\mathbb{Z}_3^\times} = \epsilon_K$, and the fact that $\chi|_{I_K}$ does not factor through the norm, imply that $\chi(-1) = \chi(u_2^3) = -1$. (Recall that we have chosen $\zeta_6 = -\zeta_3^2$ so that $\zeta_6^2 = \zeta_3$.) Since there are no other constraints, we must have $\chi(u_2) = \zeta_6^i = (-1)^i\zeta_3^{2i}$ with $i = 1, 3, 5$, which is the same as $\chi(u_2) = -\zeta_3^{2j}$ with $j = 1, 0, 2$, respectively. Thus, we have six possible choices for $\chi|_{I_K}$, giving three pairs of conjugate characters on I_K. Therefore, we can choose $\chi = \chi_{(-3,4,6)_j}$ for $j = 0, 1, 2$. Thus $\tau \simeq \tau_{\mathrm{sc},3}(-3, 4, 6)_j$. ☐

Proof (c). Suppose $N_E = 3^4$.

First suppose that τ is a principal series. Then $\tau = \chi|_{I_3} \oplus \chi^{-1}|_{I_3}$, where χ is a character of W_3 with conductor 3^2. By Lemma 5.1.1, we have that $\chi|_{I_3}$ factors through

$$(\mathbb{Z}_3/3^2\mathbb{Z}_3)^\times \simeq \langle -1 \rangle \times \langle 4 \rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$$

Since χ is primitive at this conductor, we have $\chi(4) = \zeta_3^{\pm 1}$. Twisting by ε_3, we can assume that $\chi(-1) = 1$. Thus, there are two possibilities for $\chi|_{I_3}$, which are $\chi_{(1,2,3)}$ and $\chi_{(1,2,3)}^{-1}$. Thus $\tau \simeq \tau_{\mathrm{ps},3}(1, 2, 3)$ (hence $e = 3$) or $\tau \simeq \tau_{\mathrm{ps},3}(1, 2, 3) \otimes \varepsilon_3$ (hence $e = 6$).

We are left with the case where τ is supercuspidal. In this case, since $\mathrm{condexp}(\tau) = 4$ is even, it follows that τ is an induction of a character χ from the unramified extension $K = \mathbb{Q}_3(\sqrt{-1})$ (see [7, Chap IV, § 15]). Therefore χ has conductor 3^2 and $\chi|_{I_K}$ gives a non-trivial character

$$\left(\mathcal{O}_K/(3^2)\right)^\times/U = \langle u \rangle \simeq \mathbb{Z}/12\mathbb{Z}$$

where $u = \sqrt{-1} + 2$. By Corollary 5.1.4, $\chi(4) = \chi(-1) = 1$. By primitivity, $\chi(u)$ must have order 3, 6 or 12 (Lemma 5.1.1). However, by Lemma 3.2.1, the case $e = 12$ only arises for non-abelian inertia. Thus $\chi(u)$ has order 3 or 6, meaning that $\chi(u) = \zeta_6^j$, with $j \in \{1, 2, 4, 5\}$.

Therefore, there are four possible choices for $\chi|_{I_K}$, which divide into two pairs of conjugate characters. (Note that Corollary 5.1.5 implies that χ does not factor through the norm.)

Finally, observe that $\delta = \chi \cdot (\chi_3|_K)$ is a character of W_K also of conductor 3^2 such that $\delta(u)$ has order 3 or 6, and

$$\delta(x) = \chi(x)\chi_3(\mathrm{Nm}_{K|\mathbb{Q}_3}(x)) = \chi(x)\varepsilon_3(x^2) = \chi(x) = 1, \text{ for all } x \in \mathbb{Z}_3^\times.$$

Therefore, $\delta|_{\mathcal{O}_K^\times}$ must be one of the previous four characters. Thus, up to twisting by ε_3, we can assume that $\chi(u) = \zeta_6^j$, for $j = 2, 4$. This is the same as requiring that $\chi(u) = \zeta_3^{\pm 1}$, and this gives two conjugate characters. Thus we can take $\chi|_{I_K} = \chi_{(-1,2,3)}$, and we conclude that $\tau \simeq \tau_{\mathrm{sc},3}(-1, 2, 3)$ and $e = 3$ or $\tau \simeq \tau_{\mathrm{sc},3}(-1, 2, 3) \otimes \epsilon_3$ and $e = 6$. □

Example 5.2.2. Case $\ell = 3$ for potentially good reduction e$= 1$: 11a1 (good reduction) e$=$ 2 : 99d2 $N_E = 3^2$, e=4 : 36a1 $N_E = 3^4$, e=3 : 162b1 (principal series), 162d1 (supercuspidal) $N_E = 3^4$, e=6 : 162c2 (principal series), 162a1 (supercuspidal) $N_E = 3^3$, e=12 : 320c2, 320f2 : 27a1, 54a1 $N_E = 3^5$, e=12 : 192a2, 576f2 : 243a1, 243b1, 972a1

6. Nonexceptional inertial types at $\ell = 2$

In this section, we begin our consideration of inertial types for the case $\ell = 2$. We treat all inertial types but for the exceptional supercuspidal types, leaving the latter for Section 7.2.

6.1. **Setup and statement of result.** In this section, we let $K = \mathbb{Q}_2(\sqrt{d})$ be one of the seven quadratic extensions of $\mathbb{Q}_2$, so $d = -4, 5, \pm 8, -20, \pm 40$. The unique unramified one is $\mathbb{Q}_2(\sqrt{5})/\mathbb{Q}_2$, the remaining have conductor 2^2 or 2^3. We write $\mathcal{O}_K$ for the ring of integers of K, and $\mathfrak{p}$ for the prime of K. Let $s \in G_2$ be a lift of the non-trivial element of $\mathrm{Gal}(K \mid \mathbb{Q}_2)$.

Let ε_d be the restriction to I_2 of the character of G_2 fixing $K = \mathbb{Q}_2(\sqrt{d})$, so ε_d is nontrivial except for $d = 5$. We define them explicitly:

- ε_{-4} has conductor 2^2 and is defined on $(\mathbb{Z}_2/2^2\mathbb{Z}_2)^\times$ by
$$\varepsilon_{-4}(-1) = -1.$$
- ε_8 has conductor 2^3 and is defined on $(\mathbb{Z}_2/2^3\mathbb{Z}_2)^\times$ by
$$\varepsilon_8(-1) = 1, \quad \varepsilon_8(5) = -1.$$
- ε_{-8} has conductor 2^3, satisfies $\varepsilon_{-8} = \varepsilon_{-1}\varepsilon_8$, and is defined on $(\mathbb{Z}_2/2^3\mathbb{Z}_2)^\times$ by
$$\varepsilon_{-8}(-1) = -1, \quad \varepsilon_{-8}(5) = -1.$$
- Since we are restricted to inertia, we have
$$\varepsilon_8 = \varepsilon_{40}, \quad \varepsilon_{-8} = \varepsilon_{-40}, \quad \varepsilon_{-4} = \varepsilon_{-20}.$$

Lemma 6.1.1. *Let $K/\mathbb{Q}_2$ be one of the extensions above, $\mathfrak{f} = \mathfrak{p}^k$ and $f = \mathbb{Z}_2 \cap \mathfrak{f}$, with $k \geq 1$ integer. Table 5 gives the structure and explicit generators for the group $(\mathcal{O}_K/\mathfrak{f})^\times/U$, where $U = \{1\}$ if $K = \mathbb{Q}_2$ and $U = \mathrm{Nm}_{K|\mathbb{Q}_2}((\mathcal{O}_K/\mathfrak{f})^\times)$, $\mathrm{Nm}_{K|\mathbb{Q}_2} : (\mathcal{O}_K/\mathfrak{f})^\times \to (\mathbb{Z}_2/f)^\times$, otherwise.*

Proof. The proof is by induction on k. □

Lemma 6.1.2. *Let $K/\mathbb{Q}_2$ be one of the quadratic extensions above, and $\mathfrak{f} = \mathfrak{p}^k$, with $k \geq 1$ integer. Let $U = \mathrm{Nm}_{K|\mathbb{Q}_2}((\mathcal{O}_K/\mathfrak{f})^\times)$ and $f = \mathbb{Z}_2 \cap \mathfrak{f}$. Then, we have*

(i) $(\mathbb{Z}_2/f)^\times = U$ if $K = \mathbb{Q}_2(\sqrt{c})$, for $c = \pm 5, -1$;

K	$\mathfrak{f}$	$(\mathcal{O}_K/\mathfrak{f})^\times/U$	Group structure
$\mathbb{Q}_2$	2^4	$\langle -1\rangle \times \langle 5\rangle$	$\mathbb{Z}/2\times\mathbb{Z}/4$
$\mathbb{Q}_2(\sqrt{5})$	$2^k(k=1,2)$	$\langle w-1\rangle$	$\mathbb{Z}/(3\cdot 2^{k-1})$
	$2^k(k\geq 3)$	$\langle 2w-1\rangle\times\langle w-1\rangle$	$\mathbb{Z}/2\times\mathbb{Z}/3\cdot 2^{k-2}$
$\mathbb{Q}_2(\sqrt{c})$ $c=-1,-5$	$\mathfrak{p}$	$\{0\}$	Trivial
	$\mathfrak{p}^2$	$\langle\sqrt{c}\rangle$	$\mathbb{Z}/2$
	$\mathfrak{p}^k(k\geq 3)$	$\langle\sqrt{c}\rangle\times\langle 2\sqrt{c}-1\rangle$	$\mathbb{Z}/4\times\mathbb{Z}/2^{\lfloor\frac{k-2}{2}\rfloor}$
$\mathbb{Q}_2(\sqrt{c})$ $c=\pm 2,\pm 10$	$\mathfrak{p}^k(k=1,\dots,4)$	$\langle\sqrt{c}-1\rangle$	$\mathbb{Z}/2^{\lfloor\frac{k}{2}\rfloor}$
	$\mathfrak{p}^k(k\geq 5)$	$\langle -3\rangle\times\langle\sqrt{c}-1\rangle$	$\mathbb{Z}/2\times\mathbb{Z}/2^{\lfloor\frac{k}{2}\rfloor}$

TABLE 5. The group structure of the group $(\mathcal{O}_K/\mathfrak{f})^\times/U$. (Here $w=\frac{1+\sqrt{5}}{2}$.)

(ii) $(\mathbb{Z}_2/f)^\times/U=\langle 5\rangle\simeq\mathbb{Z}/2\mathbb{Z}$ *if* $K=\mathbb{Q}_2(\sqrt{c})$, *for* $c=\pm 10,\pm 2$.

Proof. The proof uses induction on k the same way Lemma 6.1.1 does. □

Lemma 6.1.3. *Let $K/\mathbb{Q}_2$ be one of the quadratic extensions above, and $\mathfrak{f}=\mathfrak{p}^k$, with $k\geq 1$ integer. Let $U=\mathrm{Nm}_{K|\mathbb{Q}_2}((\mathcal{O}_K/\mathfrak{f})^\times)$ and $H:=\ker(\mathrm{Nm}_{K|\mathbb{Q}_2})$, and recall the exact sequence*

$$1\longrightarrow U\longrightarrow(\mathcal{O}_K/\mathfrak{f})^\times\xrightarrow{\varphi}(\mathcal{O}_K/\mathfrak{f})^\times/U\longrightarrow 1.$$

Then the image $\varphi(H)$ is given in Table 6, and the group structures of U and $H\cap U$ in Table 7.

Proof. □

Using Lemma 6.1.3, ...

Corollary 6.1.4. *Let $K/\mathbb{Q}_2$ be one of the quadratic extensions above, and $\mathfrak{f}=\mathfrak{p}^k$, with $k\geq 1$ integer. Let $U=\mathrm{Nm}_{K|\mathbb{Q}_2}((\mathcal{O}_K/\mathfrak{f})^\times)$, and $\chi:(\mathcal{O}_K/\mathfrak{f})^\times\to\mathbb{C}^\times$ a character. Then, we have*

(i) For $K=\mathbb{Q}_2(\sqrt{5})$, $\chi|_{\mathbb{Z}_2^\times}=\epsilon_K \iff \chi|_U=1 \iff \chi(4)=\chi(-1)=1$.

(ii) For $K=\mathbb{Q}_2(\sqrt{c})$, $c=-1,-5$, $\chi|_{\mathbb{Z}_2^\times}=\epsilon_K \iff \chi|_U=1 \iff \chi(4)=\chi(-1)=1$.

(iii) For $K=\mathbb{Q}_2(\sqrt{c})$, $c=2,10$, $\chi|_{\mathbb{Z}_2^\times}=\epsilon_K \iff \chi|_U=1$ *and* $\chi(-1)=-1$ $\iff \chi(4)=1$ *and* $\chi(-1)=-1$.

(iv) For $K=\mathbb{Q}_2(\sqrt{c})$, $c=-2,-10$, $\chi|_{\mathbb{Z}_2^\times}=\epsilon_K \iff \chi|_U=1$ *and* $\chi(-1)=-1$ $\iff \chi(4)=1$ *and* $\chi(-1)=-1$.

Proof. This follows from Lemmas 6.1.2 and 6.1.3. □

Corollary 6.1.5. *Let $K/\mathbb{Q}_3$ be one of the ramified quadratic extensions above, and $\mathfrak{f}=\mathfrak{p}^k$, with $k\geq 1$ integer. Let $U=\mathrm{Nm}_{K|\mathbb{Q}_3}((\mathcal{O}_K/\mathfrak{f})^\times)$, and $\chi:(\mathcal{O}_K/\mathfrak{f})^\times\to\mathbb{C}^\times$ a character. Then, χ factors through the norm map if and only if $\chi|_H$ is trivial.*

Proof. □

Next, we define the following types, with generators as in Lemma **??**.
The main result of this section is as follows.

K	$\mathfrak{f}$	$\varphi(H)$	Group structure
$\mathbb{Q}_2(\sqrt{5})$	$2^k (k = 1, 2)$	$\langle w - 1 \rangle$	$\mathbb{Z}/3$
	$2^k (k \geq 3)$	$\langle (w-1)^2 \rangle$	$\mathbb{Z}/3 \cdot 2^{k-3}$
$\mathbb{Q}_2(\sqrt{-1})$	$\mathfrak{p}$	Trivial	Trivial
	$\mathfrak{p}^2$	$\langle \sqrt{-1} \rangle$	$\mathbb{Z}/2$
	$\mathfrak{p}^3$	$\langle \sqrt{-1} \rangle$	$\mathbb{Z}/4$
	$\mathfrak{p}^4$	$\langle \sqrt{-1} \rangle \times \langle 2\sqrt{-1} - 1 \rangle$	$\mathbb{Z}/4 \times \mathbb{Z}/2$
	$\mathfrak{p}^5$	$\langle \sqrt{-1} \rangle$	$\mathbb{Z}/4$
	$\mathfrak{p}^k (k \geq 6)$	$\langle \sqrt{-1} \rangle \times \langle (2\sqrt{-1} - 1)^2 \rangle$	$\mathbb{Z}/4 \times \mathbb{Z}/2^{\lfloor \frac{k-4}{2} \rfloor}$
$\mathbb{Q}_2(\sqrt{-5})$	$\mathfrak{p}$	Trivial	Trivial
	$\mathfrak{p}^2$	$\langle \sqrt{-5} \rangle$	$\mathbb{Z}/2$
	$\mathfrak{p}^3$	$\langle \sqrt{-5} \rangle$	$\mathbb{Z}/4$
	$\mathfrak{p}^4$	$\langle \sqrt{-5} \rangle \times \langle 2\sqrt{-5} - 1 \rangle$	$\mathbb{Z}/4 \times \mathbb{Z}/2$
	$\mathfrak{p}^5$	$\langle \sqrt{-5} \rangle$	$\mathbb{Z}/4$
	$\mathfrak{p}^k (k \geq 6)$	$\langle -5 \rangle \times \langle 2\sqrt{-5} - 1 \rangle$	$\mathbb{Z}/2 \times \mathbb{Z}/2^{\lfloor \frac{k-2}{2} \rfloor}$
$\mathbb{Q}_2(\sqrt{c})$ $c = 2, 10$	$\mathfrak{p}$	Trivial	Trivial
	$\mathfrak{p}^2$	$\langle \sqrt{c} - 1 \rangle$	$\mathbb{Z}/2$
	$\mathfrak{p}^3$	$\langle 1 \rangle$	Trivial
	$\mathfrak{p}^4$	$\langle (\sqrt{c} - 1)^2 \rangle$	$\mathbb{Z}/2$
	$\mathfrak{p}^5$	$\langle -3 \rangle \times \langle (\sqrt{c} - 1)^2 \rangle$	$\mathbb{Z}/2 \times \mathbb{Z}/2$
	$\mathfrak{p}^6$	$\langle -3 \rangle \times \langle (\sqrt{c} - 1)^2 \rangle$	$\mathbb{Z}/2 \times \mathbb{Z}/4$
	$\mathfrak{p}^k (k \geq 7)$	$\langle (\sqrt{c} - 1)^2 \rangle$	$\mathbb{Z}/2^{\lfloor \frac{k-2}{2} \rfloor}$
$\mathbb{Q}_2(\sqrt{c})$ $c = -2, -10$	$\mathfrak{p}$	Trivial	Trivial
	$\mathfrak{p}^2$	$\langle \sqrt{c} - 1 \rangle$	$\mathbb{Z}/2$
	$\mathfrak{p}^3$	$\langle 1 \rangle$	Trivial
	$\mathfrak{p}^4$	$\langle (\sqrt{c} - 1)^2 \rangle$	$\mathbb{Z}/2$
	$\mathfrak{p}^k (k \geq 5)$	$\langle -3 \rangle \times \langle (\sqrt{c} - 1)^2 \rangle$	$\mathbb{Z}/2 \times \mathbb{Z}/2^{\lfloor \frac{k-2}{2} \rfloor}$

TABLE 6. The group structure of $\varphi(H)$, where $H = \ker(\mathrm{Nm}_{K|\mathbb{Q}_2})$.

Theorem 6.1.6. *Let E be an elliptic curve over $\mathbb{Q}_2$ with additive, potentially good reduction, conductor N_E, semistability defect e, and inertial type τ. Suppose that $e \neq 2, 24$. Then τ is given by one of the following cases:*

K	$\mathfrak{f}$	U	Group structure	$H\cap U$
$\mathbb{Q}_2(\sqrt{5})$	2	$\langle 1\rangle$	Trivial	Trivial
	2^2	$\langle 3\rangle$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
	$2^k(k\geq 3)$	$\langle -1\rangle\times\langle 3\rangle$	$\mathbb{Z}/2\times\mathbb{Z}/(2^{k-2})$	$\mathbb{Z}/2\times\mathbb{Z}/2$
$\mathbb{Q}_2(\sqrt{c})$ $c=2,10$	$\mathfrak{p}^k(k=1,2)$	$\langle 1\rangle$	Trivial	Trivial
	$\mathfrak{p}^k(k=3,\ldots,6)$	$\langle 7\rangle$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
	$\mathfrak{p}^k(k\geq 7)$	$\langle -1\rangle\times\langle 7\rangle$	$\mathbb{Z}/2\times\mathbb{Z}/(2^{\lfloor\frac{k-5}{2}\rfloor})$	$\mathbb{Z}/2\times\mathbb{Z}/2$
$\mathbb{Q}_2(\sqrt{c})$ $c=-1,-5$	$\mathfrak{p}^k(k=1,\ldots,4)$	$\langle 1\rangle$	Trivial	Trivial
	$\mathfrak{p}^k(k\geq 5)$	$\langle 5\rangle$	$\mathbb{Z}/(2^{\lfloor\frac{k-3}{2}\rfloor})$	$\mathbb{Z}/2$
$\mathbb{Q}_2(\sqrt{c})$ $c=-2,-10$	$\mathfrak{p}^k(k=1,2)$	$\langle 1\rangle$	Trivial	Trivial
	$\mathfrak{p}^k(k=3,\ldots,6)$	$\langle 3\rangle$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
	$\mathfrak{p}^k(k\geq 7)$	$\langle 3\rangle$	$\mathbb{Z}/(2^{\lfloor\frac{k-3}{2}\rfloor})$	$\mathbb{Z}/2$

TABLE 7. The group structures of U and $H\cap U$.

K	$\mathfrak{f}$	(d,f,r)	$\chi_{(d,f,r)}$ on generators	τ	cond(τ)
$\mathbb{Q}_2$	2^4	$(1,4,4)$	$1,\ i$	$\tau_{\mathrm{ps},2}(1,4,4)$	2^8
$\mathbb{Q}_2(\sqrt{5})$	2	$(5,1,3)$	ζ_3	$\tau_{\mathrm{sc},2}(5,1,3)$	2^2
	2^4	$(5,4,4)$	$1,\ i,\ -1,\ i$	$\tau_{\mathrm{sc},2}(5,4,4)$	2^8
$\mathbb{Q}_2(\sqrt{\pm 2})$	$\mathfrak{p}^5$	$(\pm 8,5,4)$	$-1,\ i,\ 1$	$\tau_{\mathrm{sc},2}(\pm 8,5,4)$	2^8
	$\mathfrak{p}^5$	$(\pm 8,5,2)$	$-1,\ 1,\ -1$	$\tau_{\mathrm{sc},2}(\pm 8,5,2)$	2^8
$\mathbb{Q}_2(\sqrt{\pm 10})$	$\mathfrak{p}^5$	$(\pm 40,5,4)$	$-1,\ i,\ 1$	$\tau_{\mathrm{sc},2}(\pm 40,5,4)$	2^8
	$\mathfrak{p}^5$	$(\pm 40,5,2)$	$-1,\ 1,\ -1$	$\tau_{\mathrm{sc},2}(\pm 40,5,2)$	2^8
$\mathbb{Q}_2(\sqrt{-1})$	$\mathfrak{p}^3$	$(-4,3,4)$	i	$\tau_{\mathrm{sc},2}(-4,3,4)$	2^5
	$\mathfrak{p}^6$	$(-4,6,4)$	$1,\ 1,\ i$	$\tau_{\mathrm{sc},2}(-4,6,4)$	2^8
$\mathbb{Q}_2(\sqrt{-5})$	$\mathfrak{p}^3$	$(-20,3,4)$	i	$\tau_{\mathrm{sc},2}(-20,3,4)$	2^5
	$\mathfrak{p}^6$	$(-20,6,4)$	$1,\ 1,\ i$	$\tau_{\mathrm{sc},2}(-20,6,4)$	2^8
	$\mathfrak{p}^6$	$(-20,6,4)_j$	$1,\ -1,\ i$	$\tau_{\mathrm{sc},2}(-20,6,4)_j$	2^8

TABLE 8. Inertial types for $\mathbb{Q}_2$

Case	N_E	e	τ
(a)	2^2	3	$\tau_{\mathrm{sc},2}(5,1,3)$
(b)	2^4	6	$\tau_{\mathrm{sc},2}(5,1,3)\otimes\varepsilon_{-1}$
(c)	2^5	8	$\tau_{\mathrm{sc},2}(-4,3,4),\tau_{\mathrm{sc},2}(-20,3,4)$
(d-i)	2^6	6	$\tau_{\mathrm{sc},2}(5,1,3)\otimes\varepsilon_{8},\tau_{\mathrm{sc},2}(5,1,3)\otimes\varepsilon_{-8}$
(d-ii)		8	$\tau_{\mathrm{sc},2}(-4,3,4)\otimes\varepsilon_{8},\tau_{\mathrm{sc},2}(-20,3,4)\otimes\varepsilon_{8}$
(e-i)	2^8	4	$\tau_{\mathrm{ps},2}(1,4,4),\tau_{\mathrm{ps},2}(1,4,4)\otimes\varepsilon_{-1},\tau_{\mathrm{sc},2}(5,4,4),\tau_{\mathrm{sc},2}(5,4,4)\otimes\varepsilon_{-1}$
(e-ii)		8	$\tau_{\mathrm{sc},2}(-4,6,4),\tau_{\mathrm{sc},2}(-4,6,4)\otimes\varepsilon_{8}$

In particular, τ is nonexceptional supercuspidal in all cases except when

$$e=4 \text{ and } N_E=2^8 \text{ and } \tau\simeq\tau_{\mathrm{ps},2}(1,4,4),\tau_{\mathrm{ps},2}(1,4,4)\otimes\varepsilon_{-4}$$

in which case τ is principal series.

The proof of Theorem 6.1.6 will be given in section 6.6 after treating various cases in the next few sections.

Example 6.1.7. $e = 1$: 11a1 (good reduction) $e = 2$: 176b2, 704a2, 704k2 correspond to characters -1, 2, -2 respectively $e = 3$: 20a1 $N_E = 2^4$, $e = 6$: 80b1 $N_E = 2^5$, $e = 8$: 96a1, 288a1 $N_E = 2^6$, $e = 6$: 320c2, 320f2 $N_E = 2^6$, $e = 8$: 192a2, 576f2 $N_E = 2^8$, $e = 8$: 256b1, 256c1 $N_E = 2^8$, $e = 4$: 256a1, 256d1 (supercuspidal) $N_E = 2^8$, $e = 4$: 768b1, 768h1 (principal series) Explicit curves giving the exceptional types are already given in the corresponding section.

Throughout, let E be as in Theorem 6.1.6: E is an elliptic curve over $\mathbb{Q}_2$ with additive, potentially good reduction, semistability defect $e \neq 2, 24$, conductor N_E, and inertial type $\tau := \rho_E|_{I_2}$.

6.2. **Principal series.** We begin with the relatively easy case of principal series.

Proposition 6.2.1. *Suppose that τ is a principal series.*
Then $N_E = 2^8$, $e = 4$, and $\tau \simeq \tau_{\mathrm{ps},2}(1,4,4), \tau_{\mathrm{ps},2}(1,4,4) \otimes \varepsilon_{-4}$.

Proof. We have $\mathrm{cond}(\tau) = 2^{2m}$ with $1 \leq m \leq 4$ by the formula (2.2.1) and Lemma 3.1.2. Thus $\tau = \chi|_{I_2} \oplus \chi^{-1}|_{I_2}$, where $\chi|_{I_2}$ factors through $(\mathbb{Z}_2/2^m\mathbb{Z}_2)^\times$. If $m \leq 3$, then $\chi|_{I_2}$ is at most quadratic, so $e \leq 2$, contradicting our running hypothesis; thus $m = 4$. From Table **??**, we know that $\chi|_{I_2}$ factors through

$$(\mathbb{Z}_2/2^4\mathbb{Z}_2)^\times = \langle -1 \rangle \times \langle 5 \rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/4$$

and primitivity forces $\chi(5) = \pm i$.

Twisting by ε_{-4} allows to assume $\chi(-1) = 1$. Thus $\chi|_{I_2} = \chi_{(1,4,4)}$ or $\chi^s_{(1,4,4)} = \chi^{-1}_{(1,4,4)}$. We conclude that $\tau \simeq \tau_{\mathrm{ps},2}(1,4,4)$ or $\tau \simeq \tau_{\mathrm{ps},2}(1,4,4) \otimes \varepsilon_{-4}$. ☐

6.3. **Quadratic inductions, conductor 8.** Having dealt with τ reducible, we consider in the remaining subsections inertial types induced from a quadratic extension $K \,|\, \mathbb{Q}_2$. Here, we rule out the possibility that M has conductor $8 = 2^3$.

Proposition 6.3.1. *Suppose that τ is irreducible and induced from a quadratic extension $K \,|\, \mathbb{Q}_2$. Then K is either unramified or it has conductor 2^2.*

Proof. We work by contradiction, so suppose that τ is induced from one of the quadratic extensions of conductor 2^3. More precisely, assume that $\rho_E \simeq \mathrm{Ind}_{W_K}^{W_2} \chi$ for a character $\chi \colon W_K \to \mathbb{C}^\times$, where $K = \mathbb{Q}_2(\sqrt{c})$ with $c = \pm 2, \pm 10$ (or $d = \pm 8, \pm 40$).

The condition $\chi|_{\mathbb{Z}_2^\times} = \varepsilon_d$ implies $\chi(5) = \chi(-3) = -1$ as $-3 \equiv 5 \pmod 8$. On the other hand, the conductor formula $\mathrm{condexp}(\tau) = 3 + \mathrm{condexp}(\chi)$ and Lemma 3.1.2 imply $\mathrm{condexp}(\chi) \leq 5$. Since all characters of conductor at most $\mathfrak{p}^4$ have $\chi(-3) = 1$ (Lemma **??**), we must have $\mathrm{cond}(\tau) = 2^8$ and $\mathrm{condexp}(\chi) = 5$. By Lemma **??**, $\chi|_{I_K}$ factors through

$$(\mathcal{O}_K/\mathfrak{p}^5)^\times = \langle -3 \rangle \times \langle \sqrt{c} - 1 \rangle \times \langle -1 \rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/2.$$

Observe that primitivity forces only $\chi(-3) = -1$ which we already know. To determine the value of χ on $u = \sqrt{c} - 1$, observe that

$$\chi(u)\chi^s(u) = \chi(-(\sqrt{c}-1)(\sqrt{c}+1)) = \chi(-1)\chi(c-1) = \epsilon_d(-1)\epsilon_d(c-1) = 1,$$

therefore $\chi^s(u) = \chi^{-1}(u)$ for all choices of c (or equivalently d). This forces $\chi(u) = \pm i$ as otherwise χ is quadratic and factors via the norm; in particular, $e = 8$. Moreover, for the uniformizer $\sqrt{c}$ in K, we have

$$\chi^s(\sqrt{c}) = \chi(-\sqrt{c}) = \varepsilon_d(-1)\chi(\sqrt{c})$$

and we conclude that χ/χ^s is quadratic; thus χ/χ^s factors via the norm by Corollary **??**. Therefore ρ_E is *triply imprimitive* in the language of Bushnell-Henniart [6, section 41.3]. From Gérardin [20, section 2.7], it then follows that ρ has projective image $D_2 \simeq C_2 \times C_2$.

On the other hand, by Dokchitser-Dokchitser [17, Lemma 1], there is a twist of ρ which factors through $\mathbb{Q}_2(E[3])$, so $\mathrm{P}\rho \simeq \mathrm{P}\overline{\rho}_{E,3}$. Therefore, since $e = 8$ (hence $\Phi \simeq Q_8$ is quaternion of order 8 by Lemma 3.2.1), $\overline{\rho}_{E,3}(G_{\mathbb{Q}_2}) \leq \mathrm{GL}_2(\mathbb{F}_3)$ is the 2-Sylow subgroup [17, Table 1] for which $\mathrm{P}\overline{\rho}_{E,3}(G_{\mathbb{Q}_2}) \not\simeq C_2 \times C_2$, giving a contradiction. □

6.4. **Quadratic unramified inductions.** We now consider the case of inertial types induced from the unramified quadratic extension $K = \mathbb{Q}(\sqrt{5})$.

Proposition 6.4.1. *Suppose τ is (nonexceptional) supercuspidal, obtained by inducing a character χ of W_K, where $K = \mathbb{Q}_2(\sqrt{5})$. Then τ is given by one of the following cases:*

(a) *If $N_E = 2^2$, then $e = 3$ and $\tau \simeq \tau_{\mathrm{sc},2}(5,1,3)$.*
(b) *If $N_E = 2^4$, then $e = 6$ and $\tau \simeq \tau_{\mathrm{sc},2}(5,1,3) \otimes \varepsilon_{-4}$.*
(c) *If $N_E = 2^6$, then $e = 6$ and $\tau \simeq \tau_{\mathrm{sc},2}(5,1,3) \otimes \varepsilon_8, \tau_{\mathrm{sc},2}(5,1,3) \otimes \varepsilon_{-8}$.*
(d) *If $N_E = 2^8$, then $e = 4$ and $\tau \simeq \tau_{\mathrm{sc},2}(5,4,4), \tau_{\mathrm{sc},2}(5,4,4) \otimes \varepsilon_{-4}$.*

Proof. Since $K \mid \mathbb{Q}_2$ is unramified then $I_2 \subset W_K$ and we are in case (iii) of Proposition 2.4.1. So τ has abelian image and $e = 3, 4, 6$ (we excluded $e = 2, 24$ at the start). The conductor of τ is 2^{2m}, which means that χ has conductor $\mathfrak{p}^m$ with $m \leq 4$ by Lemma 3.1.2.

In this case the determinant condition is $\chi|_{\mathbb{Z}_2^\times} = 1$; in particular, $\chi(-1) = \chi(5) = 1$.

Suppose $m = 1$; then $N_E = 2^2$, and the reduction is tame, hence $e = 3$. From Lemma **??**, we have $(\mathcal{O}_K/\mathfrak{p})^\times = \langle w \rangle \simeq \mathbb{Z}/3$, where $w = \frac{\sqrt{5}-1}{2}$. So, $\chi(w) = \zeta_3^{\pm 1}$ and χ does not factor through the norm by Corollary **??**. Thus there are two conjugated possibilites for $\chi|_{I_2}$ and we can take $\chi|_{I_2} = \chi_{(5,1,3)}$. Thus $\tau \simeq \tau_{\mathrm{sc},2}(5,1,3)$, proving (a).

Suppose $m = 2$. From Lemma **??**, $\chi|_{I_2}$ factors through

$$(\mathcal{O}_K/\mathfrak{p}^2)^\times = \langle -1 \rangle \times \langle w \rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/6.$$

and primitivity of χ implies $\chi(w) = \zeta_6^{\pm 1}$ and again, by part 3) of Corollary **??**, we conclude χ does not factor via the norm. Thus there are two conjugate choices for $\chi|_{I_2}$, giving rise to the same type τ', showing that there is a unique type of conductor 2^4. But, twisting an elliptic curve with inertial type $\tau_{\mathrm{sc},2}(5,1,3)$ by -1, gives an elliptic curve of conductor 2^4 and inertial type $\tau_{\mathrm{sc},2}(5,1,3) \otimes \varepsilon_{-4}$. Since τ' is the unique inertial type of conductor 2^4, we must have $\tau' = \tau_{\mathrm{sc},2}(5,1,3) \otimes \varepsilon_{-4}$, proving (b).

Suppose $m = 3$. From Lemma **??**, we know that $\chi|_{I_2}$ factors via

$$(\mathcal{O}_K/\mathfrak{p}^3)^\times/U = \langle u_1 \rangle \times \langle w \rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/6.$$

where $u_1 = 2w+1$, and primitivity of χ implies $\chi(u_1) = -1$. Furthermore, we have $\chi(w) = \zeta_6^j$ with $0 \leq j \leq 5$. For $u = u_1$ or $u = w$, we have

$$\chi(u)\chi^s(u) = \chi(\mathrm{Norm}(u)) = 1 \implies \chi^s(u) = \chi^{-1}(u) \tag{6.4.2}$$

and so χ does not factor through the norm if and only if $j \neq 0, 3$ (see Corollary **??**). This gives four possibilities for $\chi|_{I_2}$ yielding two pairs of conjugate characters, and hence two possible inertial types of conductor 2^6. But, twisting an elliptic curve with inertial type $\tau_{\mathrm{sc},2}(5,1,3)$ by 2 and -2 gives an elliptic curve of conductor 2^6 and inertial type $\tau_{\mathrm{sc},2}(5,1,3) \otimes \varepsilon_8$ and $\tau_{\mathrm{sc},2}(5,1,3) \otimes \varepsilon_{-8}$, respectively. So, $\tau_{\mathrm{sc},2}(5,1,3) \otimes \varepsilon_8$ and $\tau_{\mathrm{sc},2}(5,1,3) \otimes \varepsilon_{-8}$ must be the two inertial types of conductor 2^6. This proves (c).

Suppose $m = 4$. From Lemma **??**, we know that $\chi|_{I_2}$ factors via

$$(\mathcal{O}_K/\mathfrak{p}^4)^\times/U = \langle u_1 \rangle \times \langle w \rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/12$$

and primitivity of χ implies that $\chi(w)$ has order 4 or 12. In the latter case, the image of τ has size $e = 12$, a contradiction. So $\chi(w) = \pm i$. Since there are no further constraints we can have $\chi(u_1) = \pm 1$. This gives four possible characters. Observe that $\delta = \chi \cdot \varepsilon_{-4}|_K$ also has conductor $\mathfrak{p}^4$, satisfies $\delta|_{\mathbb{Z}_2^\times} = 1$ and it does not factor through the norm. Moreover, it satisfies $\delta(u_1) = -\chi(u_1)$ and since equation (6.4.2) holds for all $u \in \mathcal{O}_K^\times$ (note it depends only on the determinant restriction and not in the conductor), we conclude $\delta \neq \chi^{-1} = \chi^s$. Hence the possibilities for $\chi|_{I_2}$ are

$$\chi_{(5,4,4)},\ \chi^s_{(5,4,4)},\ \chi_{(5,4,4)} \cdot \varepsilon_{-4},\ \text{or}\ (\chi_{(5,4,4)} \cdot \varepsilon_{-4})^s,$$

therefore $\tau \simeq \tau_{\mathrm{sc},2}(5,4,4)$ or $\tau_{\mathrm{sc},2}(5,4,4) \otimes \varepsilon_{-4}$, as desired.

(Note that the twisted types $\tau_{\mathrm{sc},2}(5,4,4) \otimes \epsilon_{\pm 8}$ also have conductor 2^8 but they do not appear above due to the relations $\chi^s_{(5,4,4)} = \chi_{(5,4,4)}\varepsilon_{-8}$ and $(\chi_{(5,4,4)}\varepsilon_{-4})^s = \chi_{(5,4,4)}\varepsilon_8$.) ☐

6.5. **Quadratic inductions, conductor 4.** We conclude with the case of conductor $4 = 2^2$.

Proposition 6.5.1. *Suppose τ is (nonexceptional) supercuspidal, obtained by inducing a character χ of W_K where $K \supseteq \mathbb{Q}_2$ has conductor $4 = 2^2$. Then τ is given by one of the following cases:*

(a) *If $N_E = 2^5$, then $e = 8$ and $\tau \simeq \tau_{\mathrm{sc},2}(-4,3,4), \tau_{\mathrm{sc},2}(-20,3,4)$.*
(b) *If $N_E = 2^6$, then $e = 8$ and $\tau \simeq \tau_{\mathrm{sc},2}(-4,3,4) \otimes \varepsilon_8, \tau_{\mathrm{sc},2}(-20,3,4) \otimes \varepsilon_8$.*
(c) *If $N_E = 2^8$, then $e = 8$ and $\tau \simeq \tau_{\mathrm{sc},2}(-4,6,4), \tau_{\mathrm{sc},2}(-4,6,4) \otimes \varepsilon_8$.*

Proof. The quadratic extensions of conductor 2^2 are $K = \mathbb{Q}_2(\sqrt{c})$ with $c = -1$ or $c = -5$. In both cases, we have $\chi|_{\mathbb{Z}_2^\times} = \varepsilon_{-4}$ so $\chi(-1) = -1$ and $\chi(5) = 1$, hence χ does not factor through the norm by Corollary **??**. Moreover, the conductor formula (2.3.3) and Lemma 3.1.2 imply $\mathrm{condexp}(\chi) = m \leq 6$. Since all characters with of conductor exponent ≤ 2 satisfy $\chi(-1) = 1$ we have $3 \leq m \leq 6$.

We conclude that $\mathrm{cond}(\tau) = N_E = 2^k$ with $k = 5, 6, 8$, thus χ is of conductor $\mathfrak{p}^m$ with $m = k - 2 = 3, 4, 6$, respectively. We split in cases according to K.

Case 1. Suppose that $K = \mathbb{Q}(\sqrt{c})$ with $c = -1$; this is the case $d = -4$.

Suppose $m = 3$. By Lemma **??** and primitivity, we know that $\chi|_{I_2}$ factors through

$$(\mathcal{O}_K/\mathfrak{p}^3)^\times = \langle \sqrt{c} \rangle \simeq \mathbb{Z}/4$$

and satisfy $\chi(\sqrt{c}) = \pm i$. Thus, there are two possible conjugate choices and we can take $\chi|_{I_K} = \chi_{(-4,3,4)}$. Thus $\tau \simeq \tau_{\mathrm{sc},2}(-4,3,4)$. This proves (a) for $d = -4$.

Suppose $m = 4$. By Lemma **??**, we have that

$$(\mathcal{O}_K/\mathfrak{p}^4)^\times = \langle \sqrt{c} \rangle \times \langle 2\sqrt{c} - 1 \rangle \simeq \mathbb{Z}/4 \times \mathbb{Z}/2.$$

The previous case and primitivity implies $\chi(2\sqrt{c}-1) = -1$. Since $\chi(-1) = \chi(\sqrt{c})^2 = -1$, we have $\chi(\sqrt{c}) = \pm i$. Therefore, there are two conjugate choices, and a unique type of conductor 2^6. But, twisting an elliptic curve with inertial type $\tau_{\text{sc},2}(-4,3,4)$ by 2 gives an elliptic curve with conductor 2^6 and inertial type $\tau_{\text{sc},2}(-4,3,4) \otimes \varepsilon_8$. By the uniqueness of the type at conductor 2^6, we must have $\tau \simeq \tau_{\text{sc},2}(-4,3,4) \otimes \varepsilon_8$. This proves (b) for $d = -4$.

Finally suppose $m = 6$. By Lemma **??**, we have that

$$(\mathcal{O}_K/\mathfrak{p}^6)^\times = \langle 5 \rangle \times \langle \sqrt{c} \rangle \times \langle 2\sqrt{c}-1 \rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/4$$

and primitivity forces $\chi(2\sqrt{c}-1) = \pm i$. As in the case $m = 3$, we have $\chi(\sqrt{c}) = \pm i$. As $\chi(5) = 1$, this gives rise to four characters. Now observe that $\delta = \chi \cdot \varepsilon_8|_K$ also has conductor $\mathfrak{p}^6$, satisfies $\delta|_{\mathbb{Z}_2^\times} = \varepsilon_{-4}$, and it does not factor through the norm. Moreover, it satisfies $\delta(5) = -\chi(5)$ and since $\chi^s(5) = \chi(5)$ we have $\delta \neq \chi^s$. We conlude the possibilities for $\chi|_{I_K}$ are

$$\chi_{(-4,6,4)}, \quad \chi^s_{(-4,6,4)}, \quad \chi_{(-4,6,4)} \cdot \varepsilon_8|_K, \quad (\chi_{(-4,6,4)} \cdot \varepsilon_8|_K)^s,$$

yielding $\tau = \tau_{\text{sc},2}(-4,6,4)$ or $\tau_{\text{sc},2}(-4,6,4) \otimes \varepsilon_8$, proving (c) for $c = -1$.

Case 2. Suppose that $K = \mathbb{Q}(\sqrt{c})$ with $c = -5$; this is the case $d = -20$. For $m = 3, 4$, the same argument as for $d = -4$ applies (see Lemma **??** for group structures). This proves (a) and (b) for $d = -20$.

To conclude, suppose $m = 6$: we will show there is no inertial type in this case. We have

$$(\mathcal{O}_K/\mathfrak{p}^6)^\times = \langle 5 \rangle \times \langle \sqrt{c} \rangle \times \langle 2\sqrt{c}-1 \rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/4$$

and similarly as above, we have

$$\chi(5) = 1, \quad \chi(-1) = -1, \quad \chi(\sqrt{c}) = \pm i, \quad \chi(2\sqrt{c}-1) = \pm i.$$

Note that

$$\chi^s(2\sqrt{c}-1)\chi(2\sqrt{c}-1) = \chi(\text{Nm}_{K|\mathbb{Q}_2}(2\sqrt{c}-1)) = \chi(21) = \varepsilon_{-4}(21) = 1,$$

therefore

$$\chi^s(2\sqrt{c}-1) = -\chi(2\sqrt{c}-1), \quad \text{and} \quad \chi^s(\sqrt{c}) = \chi(-\sqrt{c}) = -\chi(\sqrt{c}).$$

We conclude that χ/χ^s restricted to inertia is quadratic and so factors via the norm on $\mathcal{O}_K^\times$. Observe that $1 \pm \sqrt{c}$ are uniformizers and we have

$$\chi^s(1+\sqrt{c})\chi(1+\sqrt{c}) = \chi\left(\text{Nm}_{K|\mathbb{Q}_2}(1+\sqrt{c})\right) = \chi(6) = \chi(u)\chi(1+\sqrt{c})^2,$$

where $u = (-d-2)/3$. Thus $\chi^s(1+\sqrt{c}) = \chi(u)\chi(1+\sqrt{c})$. Moreover, we have

$$u \equiv 5^2 \cdot \sqrt{c}^3 \cdot (2\sqrt{c}-1) \pmod{\mathfrak{p}^6}$$

hence

$$\chi(u) = \chi(5)^2\chi(\sqrt{c})^3\chi(2\sqrt{c}-1) = 1 \cdot (\pm i)^3 \cdot (\pm i) = \pm 1,$$

and we conclude χ/χ^s is a quadratic character. By Corollary **??** it factors via the norm, so ρ_E is triply imprimitive. This leads to a contradiction as in the proof of Proposition 6.3.1. We conclude there are no types arising from an elliptic curve for $m = 6$ and $d = -20$. $\square$

6.6. **Proof of theorem 6.1.6.** We are now ready to prove the main result of Section 6.

Proof of Theorem 6.1.6. Since $e \neq 24$, by Lemma 3.2.3 and Proposition 2.4.1, τ is either principal series or nonexceptional supercuspidal.

The case when τ is principal series is treated by Proposition 6.2.1. These types all have conductor 2^8, and correspond to case (e-i) of the theorem.

Next, suppose that τ is nonexceptional supercuspidal with conductor 2^8 induced from a quadratic extension M. Then, by Proposition 6.3.1, M has conductor 1 or 4, and these are covered in Propositions 6.4.1(d) and 6.5.1(c), respectively. This completes the proof of (e).

We are left with the nonexceptional supercuspidal types of conductor $\neq 2^8$. These are covered by Propositions 6.4.1(a)–(c) and 6.5.1(a)–(b). This implies parts (a)–(d) of the theorem. Finally, the last part of the statement follows from the fact that principal series only appear in case (e-i). ☐

7. Exceptional inertial types for $E/\mathbb{Q}_2$

Finally, we consider exceptional inertial types which arise only for $\ell = 2$.

7.1. **Setup and result.** Let $r = \pm 1, \pm 2$ and define the following elliptic curves over $\mathbb{Q}_2$

(7.1.1) $$E_{1,r} \colon ry^2 = x^3 + 3x + 2 \qquad \text{and} \qquad E_{2,r} \colon ry^2 = x^3 - 3x + 1.$$

These curves have potentially good reduction with semistability defect $e = 24$. We denote by $\tau_{i,r}$ the inertial type of $E_{i,r}$. We have $N = 0$ for all i, r as above. For reasons that will shortly be clear, we also set $\tau_{\mathrm{ex},1} := \tau_{1,1}$ and $\tau_{\mathrm{ex},2} := \tau_{2,-1}$. Our final result is as follows.

Theorem 7.1.2. *Let $E/\mathbb{Q}_2$ be an elliptic curve with potentially good reduction, semistability defect $e = 24$, conductor N_E and inertial type τ. Then, one of the following cases holds:*

- (a) *If $N_E = 2^3$, then $\tau \simeq \tau_{\mathrm{ex},2}$.*
- (b) *If $N_E = 2^4$, then $\tau \simeq \tau_{\mathrm{ex},2} \otimes \varepsilon_{-4}$.*
- (c) *If $N_E = 2^6$, then $\tau \simeq \tau_{\mathrm{ex},2} \otimes \varepsilon_8, \tau = \tau_{\mathrm{ex},2} \otimes \varepsilon_{-8}$.*
- (d) *If $N_E = 2^7$, then τ is isomorphic to one of $\tau_{\mathrm{ex},1}$, $\tau_{\mathrm{ex},1} \otimes \varepsilon_{-4}$, $\tau_{\mathrm{ex},1} \otimes \varepsilon_8$ or $\tau_{\mathrm{ex},1} \otimes \varepsilon_{-8}$.*

Proof. Let $K = \mathbb{Q}_2(E[3])$, $G = \mathrm{Gal}(K \,|\, \mathbb{Q}_2)$ and $L = \mathbb{Q}_2^{\mathrm{un}} K$ be the inertial field of E. From the proof of Lemma 3.2.3, we know that $G \simeq \mathrm{GL}_2(\mathbb{F}_3) \simeq \tilde{S}_4$ a double cover of $\mathrm{P}(\overline{\rho}_{E,3}) \simeq S_4$. From [1, Table 10] we see that all the $\tilde{S}_4$ extensions $K/\mathbb{Q}_2$ are of the form $K = \mathbb{Q}_2(E_{i,r}[3])$, where $E_{i,r}$ is one of the curves defined in (7.1.1). Therefore, there is a choice of i, r such that both τ and $\tau_{i,r}$ fix the extension $L/\mathbb{Q}_2^{\mathrm{un}}$. We have $\mathrm{Gal}(L \,|\, \mathbb{Q}_2^{\mathrm{un}}) \simeq \Phi \simeq \mathrm{SL}_2(\mathbb{F}_3)$ by Lemma 3.2.1, and since there is only one irreducible $\mathrm{GL}_2(\mathbb{C})$-representation of $\mathrm{SL}_2(\mathbb{F}_3)$ whose image is contained in $\mathrm{SL}_2(\mathbb{C})$, we conclude that $\tau \simeq \tau_{i,r}$.

Note that, for $r = -1, 2, -2$, the curve $E_{2,-r}$ is the quadratic twist of $E_{2,-1}$ by $-4, 8, -8$, respectively, therefore $\tau_{2,1} \simeq \tau_{\mathrm{ex},2} \otimes \varepsilon_{-4}$, $\tau_{2,-2} \simeq \tau_{\mathrm{ex},2} \otimes \varepsilon_8$ and $\tau_{2,2} \simeq \tau_{\mathrm{ex},2} \otimes \varepsilon_{-8}$. Similarly, we obtain $\tau_{1,-1} \simeq \tau_{\mathrm{ex},1} \otimes \varepsilon_{-4}$, $\tau_{1,2} \simeq \tau_{\mathrm{ex},1} \otimes \varepsilon_8$ and $\tau_{1,-2} \simeq \tau_{\mathrm{ex},1} \otimes \varepsilon_{-8}$. Finally, observe that the conductor of $E_{1,1}$ is 2^7 and that of $E_{2,-1}$ is 2^3, thus the eight types split in the 4 cases of the theorem according to their conductors. ☐

7.2. **Explicit characters.** Recall that, as in previous sections, we aim for an explicit description of the types in Theorem 7.1.2 in terms of characters. As explained in section 2.3, an exceptional type is determined by a triple (L, M, χ), where $L/\mathbb{Q}_2$ is a cubic extension, $M \mid L$ a quadratic extension and $\chi\colon W_M \to \mathbb{C}^\times$ a character such that $\chi \neq \chi^s$ where s is conjugation on $M \mid L$. Furthermore, we only need to specify χ on I_M.

We now define the inertial types τ^1 and τ^2, respectively, as the types associated with the triples (L_1, M_1, χ_1) and (L_2, M_2, χ_2) given as follows :

(i) Let $L_1 = L_2 = \mathbb{Q}_2(u)$, where u is a root of $x^3 - 2$;

(ii) Let $M_i = \mathbb{Q}_2(b_i)$ for $i = 1, 2$ be the degree 6 extensions defined by the Eisenstein polynomials

$$f_1 := x^6 - 6x^5 + 18x^4 - 32x^3 + 36x^2 - 24x + 10;$$
$$f_2 := x^6 - 198x^5 + 10728x^4 - 88434x^3 + 249264x^2 - 9882x + 918;$$

we have $L_i \subset M_i$ with M_i/L_i quadratic ramified;

(iii) Observe that b_i is a uniformizer in M_i, and $\mathcal{O}_{M_i} = \mathbb{Z}_2[b_i]$. Write $\mathfrak{p}_{M_i} = (b_i)$ for the unique prime ideal of $\mathcal{O}_{M_i}$. Define $\chi_1\colon W_{M_1} \to \mathbb{C}^\times$ to be a character of conductor $\mathfrak{p}_{M_1}^{11}$ such that $(\chi_1|_{I_{M_1}})^A$ is given by

$$\chi_1^A(u_1) = \chi_1^A(u_2) = \chi_1^A(u_4) = \chi_1^A(u_5) = -1 \quad \text{and} \quad \chi_1^A(u_3) = i,$$

where

$$(\mathcal{O}_{M_1}/\mathfrak{p}_{M_1}^{11})^\times = \langle u_1\rangle \times \langle u_2\rangle \times \langle u_3\rangle \times \langle u_4\rangle \times \langle u_5\rangle \tag{7.2.1}$$
$$\simeq \mathbb{Z}/16 \times \mathbb{Z}/4 \times \mathbb{Z}/4 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \tag{7.2.2}$$

and

$$u_1 := b_1 + 1, \quad u_2 := b_1^3 + 1, \quad u_3 := b_1^5 + 1, \quad u_4 := 2b_1 + 1, \quad u_5 := 2b_1^3 + 1.$$

We define also $\chi_2\colon W_{M_2} \to \mathbb{C}^\times$ to be a character of conductor $\mathfrak{p}_{M_2}^3$, whose restriction $\chi_2|_{I_{M_2}}$ is given by $(\chi_2|_{I_{M_2}})^A(1 + b_2) = i$.

Lemma 7.2.3. *We have $\tau_{\mathrm{ex},1} \simeq \tau^1$ and $\tau_{\mathrm{ex},2} \simeq \tau^2$.*

Proof. We will write E for $E_{1,1}$ or $E_{2,-1}$. Let τ be the inertial type of E and $(L, M, \chi|_{I_M})$ be the triple determining τ, so that $L/\mathbb{Q}_2$ is cubic (tamely) ramified, $M \mid L$ is quadratic and the Weil-Deligne representation $(\rho_E, 0)$ attached to E satisfies that $\rho_E|_{W_L}$ is a nonexceptional supercuspidal respresentation obtained by induction of χ from W_M to W_L. Furthermore, since $\rho_E(I_2) \simeq \mathrm{SL}_2(\mathbb{F}_3)$ it follows that M/L is ramified, otherwise $\rho_E(I_2)$ would not have an order 8 subgroup isomorphic to the quaternion group.

From [17, Lemma 1] there is an unramified twist $\rho = \rho_E \otimes \mu$ which factors through $K = \mathbb{Q}_2(E[3])$ and, since $\rho|_{I_2} = (\rho_E \otimes \mu)|_{I_2} = \rho_E|_{I_2}$, the inertial WD-type of $(\rho, 0)$ is also τ.

Write $K_x \subset K$ for the subfield generated by the x-coordinates of the 3-torsion points of E. From the proof Theorem 7.1.2, we have that $G := \mathrm{Gal}(K \mid \mathbb{Q}_2) \simeq \mathrm{GL}_2(\mathbb{F}_3)$ and $\mathrm{P}(\rho_E) = \mathrm{P}(\rho)$ factors via $G_x := \mathrm{Gal}(K_x \mid \mathbb{Q}_2) \simeq S_4$.

From all the above, we know that the image of ρ has order 48, $\rho|_{W_L}$ is an nonexceptional supercuspidal representation and $\rho|_{W_M} = \chi \oplus \chi^s$, where s is conjugation in $M \mid L$ and $\chi \neq \chi^s$.

Let $H \subset G$ be the subgroup fixing M. Since M is of degree 6 and $\rho|_{W_M} = \chi \oplus \chi^s$, it follows that H is cyclic of order 8. There are three order 8 cyclic subgroups of $\mathrm{GL}_2(\mathbb{F}_3)$ all belonging to the same conjugacy class, so one of them corresponds to H. Since $e = 24$, $|H| = 8$ and $M \mid \mathbb{Q}_2$ is totally ramified of degree 6, we see that χ is of order 8 but $\chi|_{I_M}$ has order 4.

Case $E = E_{2,-1}$. We have $\tau = \tau_{\mathrm{ex},2}$. From [17, Proposition 2] we obtain that $K/\mathbb{Q}_2$ is the splitting field of the Eisenstein polynomial

$$h_K = x^8 - 8x^7 + 26x^6 - 46x^5 + 50x^4 - 38x^3 + 22x^2 - 8x + 2.$$

One checks that f_2 splits completely over K. In particular, $M_2 \subset K$. In fact, one verifies that M_2 is the subfield of K fixed by H (i.e., fixed by one of the cyclic subgroups of order 8 in G). Since L_2 is the unique cubic subfield of M_2 it follows that, up to replacing M_2 by one of its two other Galois conjugated fileds inside K, we have $L = L_2 \subset M_2 = M$. To finish this case we need to show $\chi|_{I_M} = \chi_2|_{I_M}$.

Let $\mathfrak{p}_M$ and $\mathfrak{p}_L$ be the unique prime ideals in M and L, respectively. The curve E/L has conductor $\mathfrak{p}_L^5$ and $\upsilon_{\mathfrak{p}_L}(\Delta(M \mid L)) = 2$, hence χ has conductor $\mathfrak{p}_M^3$ by the conductor formula (2.3.3). We note that $(\mathcal{O}_M/\mathfrak{p}_M^3)^\times$ is generated by $b_2 + 1$, therefore $\chi(b_2 + 1) = \pm i$. This gives two choices for $\chi|_{I_M}$ which are conjugated by s, so we can take $\chi|_{I_M} = \chi_2|_{I_M}$ hence $\tau = \tau^2$, as desired.

Case $E = E_{1,1}$. We have $\tau = \tau_{\mathrm{ex},1}$. From [17, Proposition 2], we obtain that $K/\mathbb{Q}_2$ is the splitting field of the Eisenstein polynomial

$$h_K = x^8 - 28x^7 + 236x^6 - 280x^5 - 104x^4 - 392x^3 - 164x^2 - 112x + 2.$$

As above, one checks that f_1 splits completely over K, and that M_1 is the subfield of K fixed by H; also, L_1 is the unique cubic subfield of M_1 and so, up to conjugating M_1, we have $L = L_1 \subset M = M_1$. Now it remains to check $\chi|_{I_M} = \chi_1|_{I_M}$.

Let $\mathfrak{p}_M$ and $\mathfrak{p}_L$ be the primes in M and L, respectively. We have $\upsilon_{\mathfrak{p}_L}(\Delta(M \mid L)) = 6$ and the curve E/L has conductor $\mathfrak{p}_L^{17}$, hence χ has conductor $\mathfrak{p}_M^{11}$ by the conductor formula (2.3.3).

Recall from (7.2.1) that $(\mathcal{O}_M/\mathfrak{p}_M^{11})^\times$ is generated by u_1, u_2, u_3, u_4, u_5 of orders 16, 4, 4, 2, 2, respectively. Since χ is primitive, we must have $\chi(u_3) = \pm i$. Also, since χ fixes K and has order 4 on inertia, it follows that χ^2 fixes K_x and $\chi^2|_{I_M}$ has order 2. The field K_x is the splitting field of $x^4 - 4x^3 + 4x + 2$ (obtained from the 3-division polynomial of $E_{1,1}$). From local class field theory, we know that χ^2 fixes K_x if and only if it is trivial on the norm group $\mathrm{Norm}_{K_x/M}(K_x)$. Using the `Magma` routine `NormEquation` we check which u_i are norms from elements in K_x and we conclude that

$$\chi^2(u_1) = \chi^2(u_2) = \chi^2(u_4) = \chi^2(u_5) = 1, \quad \chi^2(u_3) = -1.$$

Similarly, we check which u_i are norms from K/M to conclude that $\chi(u_i) \neq 1$ for all i. It follows that

$$\chi(u_1) = \chi(u_2) = \chi(u_4) = \chi(u_5) = -1, \quad \chi(u_3) = \pm i$$

where the two options are s-conjugated characters. Therefore, we can take $\chi|_{I_M} = \chi_1|_{I_M}$ hence $\tau = \tau^1$, as desired. ☐

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