A DATABASE OF BASIC NUMERICAL INVARIANTS OF HILBERT MODULAR SURFACES

ERAN ASSAF, ANGELICA BABEI, BEN BREEN, EDGAR COSTA, JUANITA DUQUE-ROSERO,
ALEKSANDER HORAWA, JEAN KIEFFER, AVINASH KULKARNI, GRANT MOLNAR,
SAM SCHIAVONE, AND JOHN VOIGHT

Abstract. We describe algorithms for computing geometric invariants for Hilbert modular surfaces, and we report on their implementation.

Contents

1. Introduction

1.1. **Motivation.** Modular curves serve as essential motivation for the Langlands program, and they provide a continued and rich domain for mathematical study and explicit computation. As we consider possible generalizations, we move up in dimension and encounter Hilbert modular surfaces. A first step in surveying the fascinating landscape of Hilbert modular surfaces would be to organize and tabulate their basic arithmetic and geometric invariants, and to do so in a sufficiently general manner.

In this paper, we undertake this task. We design and implement algorithms to compute invariants—including cusp and elliptic cycle data, Chern and Betti numbers, arithmetic genus, and Kodaira type—and then compile data for a range of Hilbert modular surfaces. Our effort builds upon foundational work of Hirzebruch [8, 9], Hirzebruch–van der Geer [10], and van der Geer [17], who systematically computed invariants up to discriminant 500 restricted to level 1 and with the group SL_2. (There has also been substantial work in higher dimension, but continuing in level 1 and for the group SL_2: see the survey by Grundman [7].) The generalization to nontrivial level (and working both with SL_2 and GL_2^+) appearing here requires new analysis.

Our code is implemented in Magma [1] and is available online (`https://github.com/edgarcosta/hilbertmodularforms`). The associated dataset is available at (`https://github.com/edgarcosta/hilbertmodularsurfacesdata/`). We intend to include this data in the L-functions and Modular Forms Database (LMFDB) [12], to make it easy to browse and search.

1.2. **Organization.** The paper is organized as follows. After setting notation in section 2, we begin with the enumeration of cusps and their resolution using Hirzebruch–Jung continued fractions in section 3, generalizing work of Dasgupta–Kakde [3]. Next, in section 4 we turn to the enumeration of elliptic points and describe their rotation factors using the theory of embedding numbers and work of Prestel [14]. In section 5, we compute the generating series for the dimension of spaces of cusp forms. With these three main ingredients in hand, in section 6 we compute the desired numerical invariants, including Chern numbers, Betti numbers, and in many cases the Kodaira dimension. We conclude in section 7 with a description of the data tabulated, as well as directions for future work.

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2. Preliminaries

References for Hilbert modular forms include Freitag [5], van der Geer [17], and Goren [6]; for a computational introduction, see Dembélé–Voight [4].

2.1. **Basic notation.** Throughout, let F be a totally real field of degree $n := [F : \mathbb{Q}]$ with discriminant d_F, and let $\operatorname{Pl} F$ be the set of places of F. (We will eventually restrict F to be a real quadratic field, but some of our results hold in this general case.) Let $R := \mathbb{Z}_F$ be the ring of integers of F, and let $\operatorname{Cl} R$ be the class group of R with $h = h(R) := \# \operatorname{Cl} R$ its cardinality.

For a real place $v \colon F \hookrightarrow \mathbb{R}$ and $a \in F$, we write $a_v := v(a) \in \mathbb{R}$. Similarly, for $\alpha \in \mathrm{M}_2(F)$ write $\alpha_v \in \mathrm{M}_2(\mathbb{R})$ for the matrix obtained by applying v to a entrywise. In our algorithms, we fix an ordering of real places by the roots of a `polredabs` defining polynomial for F. Let $\operatorname{sgn} : F^\times \to \{\pm 1\}^n$ be the sign map. We say $a \in F^\times$ is totally positive if $a \in \ker \operatorname{sgn}$ (i.e., $a_v > 0$ for all v), and write $F^\times_{>0}$ (resp. $R^\times_{>0}$) for the group of totally positive elements of F (resp. totally positive units of R). Let $\operatorname{Cl}^+ R$ be the narrow class group of R with cardinality $h^+ = h^+(R) := \# \operatorname{Cl}^+ R$. There is an exact sequence

$$1 \to \{\pm 1\}^n / \operatorname{sgn}(R^\times) \to \operatorname{Cl}^+ R \to \operatorname{Cl} R \to 1 \tag{2.1.1}$$

so the natural surjection $\operatorname{Cl}^+ R \to \operatorname{Cl} R$ is an isomorphism if and only if there are units of R of all possible signs.

Let $\mathbf{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the upper half-plane equipped with its hyperbolic metric and let $\mathcal{H} := \mathbf{H}^n$, with the product indexed by the real places v. The group

$$\mathrm{GL}_2^+(F) := \{\gamma \in \mathrm{GL}_2(F) : \det \gamma \in F^\times_{>0}\} \tag{2.1.2}$$

acts naturally by orientation-preserving isometries on $\mathcal{H}$ via coordinatewise linear fractional transformations

$$z := (z_v)_v \mapsto \gamma z := (\gamma_v z_v)_v := \left(\frac{a_v z_v + b_v}{c_v z_v + d_v}\right)_v . \tag{2.1.3}$$

For a nonzero ideal $\mathfrak{N} \subseteq R$ and a (nonzero) fractional R-ideal $\mathfrak{b} \subset F$, we define the standard congruence subgroups of level $\mathfrak{N}$ by

$$\begin{aligned}
\Gamma_0(\mathfrak{N})_{\mathfrak{b}} &:= \begin{pmatrix} R & \mathfrak{b}^{-1} \\ \mathfrak{N}\mathfrak{b} & R \end{pmatrix} \cap \det{}^{-1}(R_{>0}^{\times}) \\
&= \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(F) : a, d \in R,\ b \in \mathfrak{b}^{-1},\ c \in \mathfrak{N}\mathfrak{b},\ \det\gamma \in R^{\times} \right\}, \\
\Gamma_1(\mathfrak{N})_{\mathfrak{b}} &:= \begin{pmatrix} 1+\mathfrak{N} & \mathfrak{b}^{-1} \\ \mathfrak{N}\mathfrak{b} & R \end{pmatrix} \cap \det{}^{-1}(R_{>0}^{\times})
\end{aligned} \tag{2.1.4}$$

so that $\Gamma_1(\mathfrak{N})_{\mathfrak{b}} \leq \Gamma_0(\mathfrak{N})_{\mathfrak{b}} \leq \mathrm{GL}_2^+(F)$. These subgroups arise naturally, since

$$\Gamma_0(1)_{\mathfrak{b}} = \mathrm{GL}^+(R \oplus \mathfrak{b}) \tag{2.1.5}$$

is the group of oriented R-module automorphisms of $R \oplus \mathfrak{b}$ and $\Gamma_0(\mathfrak{N})_{\mathfrak{b}}$ is the subgroup that stabilizes the first factor modulo $\mathfrak{N}$. Every projective R-module of rank 2 is isomorphic to $R \oplus \mathfrak{b}$ for some $\mathfrak{b}$.

For $\alpha \in F_{>0}^{\times}$, conjugation by $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right)$ defines an isomorphism

$$\Gamma_i(\mathfrak{N})_{\mathfrak{b}} \cong \Gamma_i(\mathfrak{N})_{\alpha\mathfrak{b}} \tag{2.1.6}$$

for $i = 0, 1$; so up to isomorphism we may take the fractional ideals $\mathfrak{b}$ among a choice of representatives of $\mathrm{Cl}^+ R$. (This corresponds to an oriented isomorphism $R \oplus \mathfrak{b} \cong R \oplus \alpha\mathfrak{b}$.) It is sometimes convenient to choose the indexing ideal $\mathfrak{b}$ to be integral and a multiple of the different of F.

We may also take the kernel of the determinant map on these groups, giving

$$\Gamma_i^1(\mathfrak{N})_{\mathfrak{b}} := \ker(\det|_{\Gamma_i(\mathfrak{N})_{\mathfrak{b}}}) = \Gamma_i(\mathfrak{N})_{\mathfrak{b}} \cap \mathrm{SL}_2(F) \tag{2.1.7}$$

for $i = 0, 1$ and an exact sequence of groups

$$1 \to \Gamma_i^1(\mathfrak{N})_{\mathfrak{b}} \to \Gamma_i(\mathfrak{N})_{\mathfrak{b}} \to R_{>0}^{\times} \to 1. \tag{2.1.8}$$

In view of (2.1.1), modulo scalars these groups agree when $\mathrm{Cl}^+ R = \mathrm{Cl}\, R$. In general, $\mathrm{P}\Gamma_i^1(\mathfrak{N})_{\mathfrak{b}}$ is a subgroup of $\mathrm{P}\Gamma_i(\mathfrak{N})_{\mathfrak{b}}$ of index dividing 2^{n-1}.

Remark 2.1.9. We might also consider other subgroups refining the determinant, for example, matrices whose determinant is 1 modulo $\mathfrak{N}$. These alternate families are organized by characters; we hope to pursue this in future work.

A more general definition of congruence subgroups is the following. For any subgroup $\Gamma \leq \mathrm{GL}_2^+(F)$, we denote by $\mathrm{P}\Gamma$ denote its image under the projection $\mathrm{GL}_2^+(F) \to \mathrm{PGL}_2^+(F)$. We define the full level subgroup (or principal congruence subgroup) of level $\mathfrak{N}$ by

$$\Gamma(\mathfrak{N})_{\mathfrak{b}} := \begin{pmatrix} 1+\mathfrak{N} & \mathfrak{N}\mathfrak{b}^{-1} \\ \mathfrak{N}\mathfrak{b} & 1+\mathfrak{N} \end{pmatrix} \cap \det{}^{-1}(R_{>0}^{\times}) \tag{2.1.10}$$

and say $\Gamma \leq \mathrm{GL}_2^+(F)$ is a congruence subgroup if $\mathrm{P}\Gamma$ is conjugate to a group that contains $\mathrm{P}\Gamma(\mathfrak{N})_{\mathfrak{b}}$ for some $\mathfrak{N}, \mathfrak{b}$. For the most part, we restrict attention to standard congruence subgroups in this article.

2.2. **Hilbert modular varieties.** Let $\Gamma < \mathrm{GL}_2^+(F)$ be a congruence subgroup. Via the action (2.1.3), the group Γ is a discrete group acting properly on $\mathcal{H}$. The group $\mathrm{P}\Gamma$ acts faithfully on $\mathcal{H}$. The quotient

$$Y(\Gamma)(\mathbb{C}) := \Gamma\backslash\mathcal{H} \tag{2.2.1}$$

is a complex orbifold of dimension n. The complex analytic space $Y(\Gamma)(\mathbb{C})$ is the set of complex points of a quasi-projective variety $Y(\Gamma)$, called a Hilbert modular variety; this variety has a canonical model, defined over its reflex field (a number field). Hilbert modular varieties admit an interpretation as a moduli space for polarized abelian surfaces with real multiplication and level structure.

The Baily–Borel compactification of $Y(\Gamma)$ is obtained by adding finitely many points as follows. Let $\mathcal{H}^* := \mathcal{H} \cup \mathbb{P}^1(F)$ be the completed product of upper half-planes. The action (2.1.3) of Γ on $\mathcal{H}$ extends to an action on $\mathcal{H}^*$. The cusps of Γ are the orbits of $\mathbb{P}^1(F)$ under Γ. Then the analytic space

$$\overline{Y}(\Gamma)(\mathbb{C}) := \Gamma\backslash\mathcal{H}^* \tag{2.2.2}$$

obtained from adding cusps is compact, and $\overline{Y}(\Gamma)$ is a proper variety. However, $\overline{Y}(\Gamma)$ is singular at the cusps when $n > 1$. The minimal desingularization

$$\pi\colon X(\Gamma) \to \overline{Y}(\Gamma) \tag{2.2.3}$$

can be understood explicitly when $n = 2$: algorithms for enumerating and resolving cusps are investigated in section 3 and the resolution of the cyclic quotient singularities is discussed in section 4.

For these varieties we will also use the notation $X_0(\mathfrak{N})_{\mathfrak{b}} = X(\Gamma_0(\mathfrak{N})_{\mathfrak{b}})$, and

$$X_0(\mathfrak{N}) := \bigsqcup_{[\mathfrak{b}]\in \mathrm{Cl}^+ R} X_0(\mathfrak{N})_{\mathfrak{b}}, \quad X_1(\mathfrak{N}) := \bigsqcup_{[\mathfrak{b}]\in \mathrm{Cl}^+ R} X_1(\mathfrak{N})_{\mathfrak{b}}, \tag{2.2.4}$$

gathering components.

2.3. **Hilbert modular forms.** Hilbert modular varieties can be understood explicitly by studying their modular forms, defined as follows. Let $k = (k_i)_i \in 2\mathbb{Z}_{\geq 0}^n$. A Hilbert modular form of (even) weight k for Γ is a holomorphic function $f\colon \mathcal{H} \to \mathbb{C}$ such that

$$f(\gamma z) = \left(\prod_v \frac{(c_v z_v + d_v)^{k_v}}{\det(\gamma_v)^{k_v/2}}\right) f(z) \tag{2.3.1}$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$, with the additional condition that f remains bounded in vertical strips in the case $F = \mathbb{Q}$.

The $\mathbb{C}$-vector space of Hilbert modular forms for $\Gamma_{\mathfrak{b}}$ of weight k is finite-dimensional and denoted $M_k(\Gamma)$. Restricting to parallel weights (i.e., all k_i are equal to some $k \in \mathbb{Z}$), multiplication gives

$$M(\Gamma) := \bigoplus_{k\in 2\mathbb{Z}_{\geq 0}} M_k(\Gamma) \tag{2.3.2}$$

the structure of a graded ring.

A cusp form is a Hilbert modular form f such that $f(z) \to 0$ as $z \to c$ for all cusps c of $\Gamma_{\mathfrak{b}}$. Let $S_k(\Gamma) \subseteq M_k(\Gamma)$ denote the subspace of cusp forms. There is a natural integration pairing, called the Petersson inner product, between $M_k(\Gamma)$ and $S_k(\Gamma)$ which provides an orthogonal decomposition

$$M_k(\Gamma) = E_k(\Gamma_{\mathfrak{b}}) \oplus S_k(\Gamma_{\mathfrak{b}}); \tag{2.3.3}$$

we call $E_k(\Gamma)$ the Eisenstein subspace.

For standard congruence subgroups, one can define related spaces of modular forms by gathering components: we write

$$\begin{aligned} M_k(\Gamma_0(\mathfrak{N})) &:= \bigoplus_{[\mathfrak{b}] \in \mathrm{Cl}^+ R} M_k(\Gamma_0(\mathfrak{N})_{\mathfrak{b}}), \\ M(\Gamma_0(\mathfrak{N})) &:= \bigoplus_{k \in 2\mathbb{Z}_{\geq 0}} M_k(\Gamma_0(\mathfrak{N})), \end{aligned} \tag{2.3.4}$$

and define similarly for $S_k(\Gamma_0(\mathfrak{N}))$, $M_k(\Gamma_0^1(\mathfrak{N}))$, etc. Multiplication also gives $M(\Gamma_0(\mathfrak{N}))$ a ring structure.

3. Cusps

A fundamental invariant of a Hilbert modular variety is the number of its cusps, the orbits of $\mathbb{P}^1(F)$ under the attached congruence subgroup Γ, arising in its compactification: this number is equal to $\dim E_k(\Gamma)$ for even $k \geq 4$. Going further, we wish to enumerate these cusps explicitly. These cusps are singular points (except for modular curves); for certain Hilbert modular surfaces, the number of curves in the minimal resolution, as well as their intersection numbers, are pleasantly determined by Hirzebruch–Jung continued fractions.

3.1. **Cusp enumeration.**

For the groups $\Gamma_1(\mathfrak{N})_{\mathfrak{b}}$, Dasgupta–Kakde [3] give an explicit enumeration of cusps. We now recall their notation and method [3, Section 3], and we generalize it to the cases of $\Gamma_0(\mathfrak{N})_{\mathfrak{b}}, \Gamma_0^1(\mathfrak{N})_{\mathfrak{b}}$, and $\Gamma_1^1(\mathfrak{N})_{\mathfrak{b}}$. In this section, F is any totally real field.

Definition 3.1.1. For $i = 0$ or 1, consider a cusp of $\Gamma_i(\mathfrak{N})_{\mathfrak{b}}$ or $\Gamma_i^1(\mathfrak{N})_{\mathfrak{b}}$ represented by $(a : c) \in \mathbb{P}^1(F)$. Define:

$$\begin{aligned} \mathfrak{s} = \mathfrak{s}(a,c) &:= aR + c\mathfrak{b}^{-1} \\ \mathfrak{M} = \mathfrak{M}(a,c) &:= \mathfrak{N} + c(\mathfrak{b}\mathfrak{s})^{-1}. \end{aligned} \tag{3.1.2}$$

Lemma 3.1.3. *The ideal $\mathfrak{M}$ is well-defined (independent of the chosen representative of the cusp) and integral.*

Proof. Scaling a, c by an element of $F^\times$ changes $\mathfrak{s}$ but this cancels out in $\mathfrak{M}$. Since $c \in \mathfrak{b}\mathfrak{s}$, the ideal $\mathfrak{M}$ is integral. $\square$

For a fractional ideal $\mathfrak{s}$ and an integral ideal $\mathfrak{M}$ of R, let $(\mathfrak{s}/\mathfrak{s}\mathfrak{M})^\times$ be the set of generators of $\mathfrak{s}/\mathfrak{s}\mathfrak{M}$ as an $(R/\mathfrak{M})$-module, and let

$$\mathcal{R}_{\mathfrak{M}}^{\mathfrak{s}} := (\mathfrak{s}/\mathfrak{s}\mathfrak{M})^\times / R_{>0}^\times, \quad \mathcal{R}_{\mathfrak{M}}^{\mathfrak{s},1} := (\mathfrak{s}/\mathfrak{s}\mathfrak{M})^\times / R^{\times 2} \tag{3.1.4}$$

be the quotients of this set under multiplication by totally positive units and under multiplication by squares of units, respectively. Let

$$\mathcal{P}_1(\mathfrak{N})_{\mathfrak{b}} := \left\{(\mathfrak{s}, \mathfrak{M}, (a,c)) : \mathfrak{N} \subseteq \mathfrak{M} \subseteq R, (a,c) \in \mathcal{R}^{\mathfrak{s}}_{\mathfrak{M}} \times \mathcal{R}^{\mathfrak{s}\mathfrak{b}\mathfrak{M}}_{\mathfrak{N}/\mathfrak{M}}\right\}. \tag{3.1.5}$$

where $\mathfrak{s}$ is a fractional ideal of F and $\mathfrak{M}$ is an integral ideal.

To deal with level $\Gamma_0(\mathfrak{N})_{\mathfrak{b}}$, we further denote

$$\mathcal{R}_{\mathfrak{M},\mathfrak{s},\mathfrak{b},\mathfrak{N}} := \left(\mathcal{R}^{\mathfrak{s}}_{\mathfrak{M}} \times \mathcal{R}^{\mathfrak{s}\mathfrak{b}\mathfrak{M}}_{\mathfrak{N}/\mathfrak{M}}\right)/(R/\mathfrak{N})^\times, \tag{3.1.6}$$

where the action of $(R/\mathfrak{N})^\times$ on $\mathcal{R}^{\mathfrak{s}}_{\mathfrak{M}} \times \mathcal{R}^{\mathfrak{s}\mathfrak{b}\mathfrak{M}}_{\mathfrak{N}/\mathfrak{M}}$ is given by

$$x \cdot (a,c) = (xa, x^{-1}c), \tag{3.1.7}$$

and similarly define

$$\mathcal{P}_0(\mathfrak{N})_{\mathfrak{b}} := \{(\mathfrak{s}, \mathfrak{M}, (a,c)) : \mathfrak{N} \subseteq \mathfrak{M} \subseteq R, (a,c) \in \mathcal{R}_{\mathfrak{M},\mathfrak{s},\mathfrak{b},\mathfrak{N}}\}. \tag{3.1.8}$$

We then have the following.

Lemma 3.1.9. *There are natural bijections*

$$\begin{aligned} \varphi_1 &: \Gamma_1(\mathfrak{N})_{\mathfrak{b}} \backslash (F^2 \setminus \{0\}) \to \mathcal{P}_1(\mathfrak{N})_{\mathfrak{b}} \\ \varphi_0 &: \Gamma_0(\mathfrak{N})_{\mathfrak{b}} \backslash (F^2 \setminus \{0\}) \to \mathcal{P}_0(\mathfrak{N})_{\mathfrak{b}} \end{aligned} \tag{3.1.10}$$

given by $(a,c) \mapsto (\mathfrak{s}, \mathfrak{M}, (\bar{a}, \bar{c}))$*, where* $\mathfrak{s} = \mathfrak{s}(a,c)$ *and* $\mathfrak{M} = \mathfrak{M}(a,c)$ *are as in* (3.1.2)*.*

Proof. The map φ_1 is well-defined and bijective by Dasgupta–Kakde [3, Lemma 3.6]. To see that φ_0 is well-defined, let $\gamma = \left(\begin{smallmatrix} p & q \\ r & t \end{smallmatrix}\right) \in \Gamma_0(\mathfrak{N})_{\mathfrak{b}}$, so that $p, t \in R$, $q \in \mathfrak{b}^{-1}$, and $r \in \mathfrak{b}\mathfrak{N}$. Then

$$\begin{aligned} \mathfrak{s}(pa+qc, ra+tc) &= (pa+qc)R + (ra+tc)\mathfrak{b}^{-1} \\ &= a(pR + r\mathfrak{b}^{-1}) + c(qR + t\mathfrak{b}^{-1}) \subseteq aR + c\mathfrak{b}^{-1} = \mathfrak{s}(a,c). \end{aligned} \tag{3.1.11}$$

Using γ^{-1}, we obtain the reverse inclusion, so $\mathfrak{s}(pa+qc, ra+tc) = \mathfrak{s}(a,c)$. Similarly,

$$\begin{aligned} \mathfrak{M}(pa+qc, ra+tc) &= \mathfrak{N} + (ra+tc)(\mathfrak{b}\mathfrak{s})^{-1} \\ &= \mathfrak{N} + ra(a\mathfrak{b} + cR)^{-1} + tc(\mathfrak{b}\mathfrak{s})^{-1} \subseteq \mathfrak{N} + c(\mathfrak{b}\mathfrak{s})^{-1} = \mathfrak{M}(a,c), \end{aligned} \tag{3.1.12}$$

and using γ^{-1}, we have $\mathfrak{M}(pa+qc, ra+tc) = \mathfrak{M}(a,c)$.

Since $q \in \mathfrak{b}^{-1}$ and $r \in \mathfrak{b}\mathfrak{N}$, we see that $qc \in c\mathfrak{b}^{-1} \subseteq \mathfrak{s}\mathfrak{M}$ and $ra \in a\mathfrak{b}\mathfrak{N} \subseteq \mathfrak{s}\mathfrak{b}\mathfrak{N}$, hence $(\overline{pa+qc}, \overline{ra+tc}) = (\overline{pa}, \overline{tc})$, where bars indicate classes in $\mathcal{R}_{\mathfrak{M},\mathfrak{s},\mathfrak{b},\mathfrak{N}}$. We also know that $pt - qr = \det(\gamma) \in R^\times_{>0}$. In particular, as $aqr \in \mathfrak{s}\mathfrak{N} \subseteq \mathfrak{s}\mathfrak{M}$, we see that

$$\overline{a} = \overline{(pt-qr)a} = \overline{pta - qra} = \overline{pta}. \tag{3.1.13}$$

Since t maps to $(R/\mathfrak{N})^\times$, its action yields $t \cdot (\overline{pa}, \overline{tc}) = (\overline{pta}, \overline{c}) = (\overline{a}, \overline{c})$, showing that the pairs are equivalent under the action, and establishing that φ_0 is well-defined.

We now show that φ_0 is bijective. Surjectivity follows from the surjectivity of φ_1. For injectivity, suppose that

$$\varphi_0(a,c) = \varphi_0(a',c') = (\mathfrak{s}, \mathfrak{M}, (\bar{a}, \bar{c})). \tag{3.1.14}$$

From the third component of (3.1.14), there exist an element $d \in (R/\mathfrak{N})^\times$ and totally positive units $\varepsilon_a, \varepsilon_c \in R^\times_{>0}$ such that

$$\begin{aligned} a' &\equiv d^{-1}\varepsilon_a a \pmod{\mathfrak{s}\mathfrak{M}} \\ c' &\equiv d\varepsilon_c c \pmod{\mathfrak{s}\mathfrak{b}\mathfrak{N}}. \end{aligned} \tag{3.1.15}$$

Choose $\delta = \left(\begin{smallmatrix} p & q \\ r & t \end{smallmatrix}\right) \in \Gamma_0(\mathfrak{N})_{\mathfrak{b}}$ such that $p \equiv d^{-1} \bmod \mathfrak{N}$ and $t \equiv d \bmod \mathfrak{N}$. Acting on (a', c') by δ, we may suppose that $a' \equiv \varepsilon_a a \pmod{\mathfrak{s}\mathfrak{M}}$ and $c' \equiv \varepsilon_c c \pmod{\mathfrak{s}\mathfrak{b}\mathfrak{N}}$. The result now follows from the injectivity of φ_1. ☐

Analogously, we define $\mathcal{R}^1_{\mathfrak{M},\mathfrak{s},\mathfrak{b},\mathfrak{N}}$ and $\mathcal{P}^1_i(\mathfrak{N})_{\mathfrak{b}}$ for $i = 0, 1$ to deal with levels $\Gamma^1_i(\mathfrak{N})_{\mathfrak{b}}$, and obtain analogous bijections

$$\begin{aligned} \varphi^1_1 &: \Gamma^1_1(\mathfrak{N})_{\mathfrak{b}} \backslash (F^2 \setminus \{0\}) \to \mathcal{P}^1_1(\mathfrak{N})_{\mathfrak{b}} \\ \varphi^1_0 &: \Gamma^1_0(\mathfrak{N})_{\mathfrak{b}} \backslash (F^2 \setminus \{0\}) \to \mathcal{P}^1_0(\mathfrak{N})_{\mathfrak{b}} \end{aligned} \tag{3.1.16}$$

due to the exact sequence $0 \to P\Gamma^1_i(\mathfrak{N})_{\mathfrak{b}} \to P\Gamma_i(\mathfrak{N})_{\mathfrak{b}} \to R^{\times}_{>0}/R^{\times 2} \to 0$.

For each $\mathfrak{M} \mid \mathfrak{N}$, we denote by $\mathfrak{Q}_i(\mathfrak{M}, \mathfrak{N})$ the set of cusps $(a : c)$ of $X_i(\mathfrak{N})$ such that $\mathfrak{M}(a, c) = \mathfrak{M}$, and by $\mathcal{P}_i(\mathfrak{M}, \mathfrak{N})_{\mathfrak{b}} \subseteq \mathcal{P}_i(\mathfrak{N})_{\mathfrak{b}}$ the set of tuples whose second coordinate is $\mathfrak{M}$, for $i \in \{0, 1\}$. We further denote by $Q_0(\mathfrak{M}, \mathfrak{N})$ the quotient of the product of narrow ray class groups $\mathrm{Cl}^+(\mathfrak{M}) \times \mathrm{Cl}^+(\mathfrak{N}/\mathfrak{M})$ by the subgroup generated by

$$\left\{ \left([xR], [x^{-1}R] \right) : x \in R,\ xR + \mathfrak{N} = R \right\}. \tag{3.1.17}$$

Similarly, we denote by $Q_1(\mathfrak{M}, \mathfrak{N})$ the quotient of $\mathrm{Cl}^+(\mathfrak{M}) \times \mathrm{Cl}^+(\mathfrak{N}/\mathfrak{M})$ by the subgroup generated by

$$\left\{ \left([xR], [x^{-1}R] \right) : x \in R,\ x \equiv 1 \pmod{\mathfrak{N}} \right\}. \tag{3.1.18}$$

We then have the following corollary.

Corollary 3.1.19. *For each $\mathfrak{M} \mid \mathfrak{N}$ and $i = 0, 1$ we have $\#\mathfrak{Q}_i(\mathfrak{M}, \mathfrak{N}) = \#Q_i(\mathfrak{M}, \mathfrak{N})$, hence the number of cusps of $X_i(\mathfrak{N})$ is $\sum_{\mathfrak{M} \mid \mathfrak{N}} \#Q_i(\mathfrak{M}, \mathfrak{N})$.*

Proof. For $i = 1$, this statement is [3, Corollary 3.12]. We will prove it for $i = 0$. By Lemma 3.1.9, taking a quotient by the natural action of $F^{\times}$ on both sides, we obtain a natural bijection

$$\mathfrak{Q}_0(\mathfrak{M}, \mathfrak{N}) \to \bigsqcup_{\mathfrak{b} \in \mathrm{Cl}^+ R} \mathcal{P}_0(\mathfrak{M}, \mathfrak{N})_{\mathfrak{b}} / F^{\times}. \tag{3.1.20}$$

There is a surjective map

$$\begin{aligned} \mathcal{P}_0(\mathfrak{M}, \mathfrak{N})_{\mathfrak{b}} / F^{\times} &\to \mathrm{Cl}\, R \\ (\mathfrak{s}, (a, c)) &\mapsto [\mathfrak{s}]. \end{aligned} \tag{3.1.21}$$

Let $\mathcal{U}$ be the image of $R^{\times}$ mapped diagonally to $\mathcal{V}(\mathfrak{M}, \mathfrak{N}) := \left(\mathcal{R}^R_{\mathfrak{M}} \times \mathcal{R}^R_{\mathfrak{N}/\mathfrak{M}} \right) / (R/\mathfrak{N})^{\times}$, where the quotient is by the action $x \mapsto (x, x^{-1})$. Then the fiber over a point in the above map is a principal homogeneous space for $\mathcal{V}(\mathfrak{M}, \mathfrak{N})/\mathcal{U}$, which is independent of $\mathfrak{b}$. Hence

$$\#\mathfrak{Q}_0(\mathfrak{M}, \mathfrak{N}) = (h^+ h) \cdot \#(\mathcal{V}(\mathfrak{M}, \mathfrak{N})/\mathcal{U}). \tag{3.1.22}$$

We can now conclude using the exact sequence

$$1 \to \mathcal{V}(\mathfrak{M}, \mathfrak{N})/\mathcal{U} \to Q_0(\mathfrak{M}, \mathfrak{N}) \to \mathrm{Cl}^+ R \times \mathrm{Cl}\, R \to 1. \tag{3.1.23}$$

☐

The above proof also implies the following corollary.

Corollary 3.1.24. *The number of cusps of* $X_0(\mathfrak{N})$ *is*

$$h^+h\sum_{\mathfrak{M}|\mathfrak{N}}\varphi_{>0}(\mathfrak{M}+\mathfrak{N}/\mathfrak{M}), \tag{3.1.25}$$

where

$$\varphi_{>0}(\mathfrak{M}) = \#\big((R/\mathfrak{M})^\times/R^\times_{>0}\big). \tag{3.1.26}$$

Proof. Consider the exact sequence

$$(R/\mathfrak{N})^\times \to (R/\mathfrak{M})^\times \times (R/(\mathfrak{N}/\mathfrak{M}))^\times \to (R/(\mathfrak{M}+\mathfrak{N}/\mathfrak{M}))^\times \to 1, \tag{3.1.27}$$

with the maps $r \mapsto (r, r^{-1})$ and $(r, s) \mapsto rs$. Recalling (3.1.22), we see that

$$\mathcal{V}(\mathfrak{M}, \mathfrak{N}) \simeq (R/(\mathfrak{M}+\mathfrak{N}/\mathfrak{M}))^\times/R^\times_{>0}, \tag{3.1.28}$$

and this isomorphism identifies $\mathcal{U}$ with squares of units. Since these are already totally positive, we deduce that the action of $\mathcal{U}$ is trivial. (This could also be observed directly using weak approximation.) The result follows. ☐

We define analogously $\mathcal{P}^1_i(\mathfrak{M}, \mathfrak{N})$, $Q^1_i(\mathfrak{M}, \mathfrak{N})$ and $\varphi^1(\mathfrak{M}) = \#\left((R/\mathfrak{M})^\times/R^{\times 2}\right)$. It follows that the number of cusps of $X^1_i(\mathfrak{N})$ is $\sum_{\mathfrak{M}|\mathfrak{N}}\#Q^1_i(\mathfrak{M}, \mathfrak{N})$. In particular, the number of cusps of $X^1_0(\mathfrak{N})$ is given by $h^+h\sum_{\mathfrak{M}|\mathfrak{N}}\varphi^1\left(\mathfrak{M}+\mathfrak{N}/\mathfrak{M}\right)$.

3.2. **Explicit computation of cusps.**

In our Magma implementation, we make two modifications to the enumeration method of section 3.1 for computational convenience. First, given $\mathfrak{s}$ and $\mathfrak{M}$, we choose a generator g for $\mathfrak{s}/\mathfrak{s}\mathfrak{M}$ as an $R/\mathfrak{M}$ module and work via the isomorphism

$$\begin{aligned} R/\mathfrak{M} &\xrightarrow{\sim} \mathfrak{s}/\mathfrak{s}\mathfrak{M} \\ r &\mapsto gs. \end{aligned} \tag{3.2.1}$$

We compute such a generator g by ensuring that $g \notin \mathfrak{s}\mathfrak{p}$ for all prime ideals $\mathfrak{p} \mid \mathfrak{M}$, which is possible by weak approximation. Second, instead of forming the sets $\mathcal{R}^{\mathfrak{s}}_{\mathfrak{M}}$ and $\mathcal{R}^{\mathfrak{s},1}_{\mathfrak{M}}$ (involving the quotient by the action of $R^\times_{>0}$ or $R^{\times 2}$) and then further taking the quotient by $F^\times$ as in (3.1.20), we consider only representatives of $\operatorname{Cl} R$ as ideals $\mathfrak{s}$ (thus accounting for the action of all units in $F^\times$ except for those in $R^\times$), and compute the remaining quotients all at once.

We are thus led to the following algorithm for computing $\mathcal{P}_i(\mathfrak{M}, \mathfrak{N})_{\mathfrak{b}}/F^\times$, where $i \in \{0, 1\}$. We use the following notation: ε is a fundamental unit of R, and ε^+ is a fundamental totally positive unit.

1. Form the direct product $D = (R/\mathfrak{M}) \times (R/(\mathfrak{N}/\mathfrak{M}))$.
2. Split according to cases.
   a. If $i = 1$, quotient $D^\times$ by the diagonal action of $R^\times_{>0}$ in each coordinate, as well as the diagonal action of $R^\times$. (Quotienting by $R^\times$ partially deals with the action of $F^\times$ mentioned above.) Explicitly, let

   $$H := \langle(\varepsilon, \varepsilon), (-1, -1), (\varepsilon^+, 1), (1, \varepsilon^+)\rangle \subseteq D^\times,$$

   and compute a transversal (i.e., a complete set of coset representatives) T for H in $D^\times$.

   b. If $i = 0$, quotient $D^\times$ by same actions as above, as well as the action by $(R/\mathfrak{N})^\times$ given in (3.1.7). Explicitly, let H be the subgroup of $D^\times$ generated by

   $$\{(\varepsilon, \varepsilon), (-1, -1), (\varepsilon^+, 1), (1, \varepsilon^+)\} \cup \{(r, r^{-1}) : r \text{ is a generator of } (R/\mathfrak{N})^\times\}$$

   and compute a transversal T for H in $D^\times$.

3. For each $[\mathfrak{s}] \in \operatorname{Cl} R$, compute generators g_1, g_2 for $\mathfrak{s}/\mathfrak{s}\mathfrak{M}$ and $\mathfrak{s}\mathfrak{b}\mathfrak{M}/\mathfrak{s}\mathfrak{b}\mathfrak{N}$ as described above. Let

   $$Q_{\mathfrak{s},\mathfrak{M}} := \{(\mathfrak{s}, \mathfrak{M}, (g_1 t_1, g_2 t_2)) : (t_1, t_2) \in T\}.$$

4. Return $\mathcal{P}_i(\mathfrak{N})_{\mathfrak{b}}/F^\times$ as $\displaystyle\bigcup_{[\mathfrak{s}] \in \operatorname{Cl} R} \bigcup_{\mathfrak{M} | \mathfrak{N}} Q_{\mathfrak{s},\mathfrak{M}}$.

We can find the cusp corresponding to a tuple $(\mathfrak{s}, \mathfrak{M}, (\overline{a}, \overline{c})) \in \mathcal{P}_i(\mathfrak{N})_{\mathfrak{b}}/F^\times$ by inverting the bijections given in Lemma 3.1.9. To do so, we seek to find $a, c \in R$ that are congruent to $\overline{a}, \overline{c}$ modulo $\mathfrak{s}\mathfrak{M}$ and $\mathfrak{s}\mathfrak{b}\mathfrak{N}$, respectively, such that $\gcd(c, \mathfrak{s}\mathfrak{b}\mathfrak{N}) = \mathfrak{s}\mathfrak{b}\mathfrak{M}$ and $\gcd(a, c\mathfrak{b}^{-1}) = \mathfrak{s}$. We first arbitrarily lift $\overline{a}$ and $\overline{c}$ to $a_0 \in \mathfrak{s}$ and $c_0 \in \mathfrak{s}\mathfrak{b}\mathfrak{M}$. Adding an element of $\mathfrak{s}\mathfrak{b}\mathfrak{N}$ to c_0 will not change the $\gcd(c_0, \mathfrak{s}\mathfrak{b}\mathfrak{N})$, so we may take $c = c_0$. Let

$$B := \{\mathfrak{p} \text{ prime ideal of } R : \mathfrak{p} \mid c\mathfrak{b}^{-1} \text{ and } \mathfrak{p} \nmid \mathfrak{s}\mathfrak{M}\}. \tag{3.2.2}$$

To find a, we construct an element $x \in R$ such that

- $x \in \mathfrak{s}\mathfrak{M}$,
- if $\mathfrak{p} \in B$ and $\mathfrak{p} \mid a_0$, then $v_{\mathfrak{p}}(x) = v_{\mathfrak{p}}(\mathfrak{s})$, and
- if $\mathfrak{p} \in B$ and $\mathfrak{p} \nmid a_0$, then $v_{\mathfrak{p}}(x) > v_{\mathfrak{p}}(\mathfrak{s})$,

and then set $a = a_0 + x$. (This again amounts to specifying valuations of x at finitely many primes.) The first condition ensures that a has the same reduction modulo $\mathfrak{s}\mathfrak{M}$ as a_0, and the latter two conditions guarantee that $\gcd(a, c\mathfrak{b}^{-1}) = \mathfrak{s}$.

Similarly, we compute $\mathcal{P}_i^1(\mathfrak{M}, \mathfrak{N})_{\mathfrak{b}}/F^\times$ using the above algorithm, replacing ε^+ by ε^2, and find the cusps corresponding to tuples in $\mathcal{P}_i^1(\mathfrak{N})_{\mathfrak{b}}/F^\times$ in the same manner.

3.3. **Resolving cusps.** Cusps usually are very singular points of the Baily–Borel compactification $\overline{Y}(\Gamma)$. In this section, we explicitly resolve cusps following [17, chapter II] and [10, chapter II.1]) assuming that F is a real quadratic field. Hirzebruch's method readily applies to full level (principal congruence) subgroups $\Gamma(\mathfrak{N})_{\mathfrak{b}}$, and we explain how to generalize it to other natural congruence subgroups. In the end, we obtain a description of the preimage of each cusp in the minimal desingularization $X(\Gamma) \to \overline{Y}(\Gamma)$ as a cyclic configuration of $\mathbb{P}^1$'s with known self-intersection numbers. This data plays an important role in the computation of the geometric invariants of the surface; cf. section 6.

We start by describing the general procedure for desingularizing cusps, for any congruence group $\Gamma < \mathrm{GL}_2^+(F)$. Let $(a : c) \in \mathbb{P}^1(F)$ be a cusp of $Y(\Gamma)$. There exists a matrix $\gamma \in \mathrm{GL}_2^+(F)$ taking $(a : c)$ to $\infty = (1 : 0)$. The resolution of $(a : c)$ can then be described in terms of the isotropy group of $\gamma\Gamma\gamma^{-1}$ at ∞, in other words, the subgroup of upper-triangular matrices in $\gamma\Gamma\gamma^{-1}$, seen as a transformation group on $\mathcal{H}$. Hirzebruch's method directly applies when this transformation group can be written as

$$G(M, V) := \begin{pmatrix} V & M \\ 0 & 1 \end{pmatrix}, \tag{3.3.1}$$

where $V \subseteq F_{>0}^\times$ is a group of totally positive units and $M \subset F$ is a $\mathbb{Z}$-module of rank 2 such that $VM = M$. We say that the cusp $(a : c)$ is of `type` $G(M, V)$.

This condition is known to hold when $\Gamma = \Gamma(\mathfrak{N})_{\mathfrak{b}}$; we will see that it also holds when Γ is one of $\Gamma_0(\mathfrak{N})_{\mathfrak{b}}$, $\Gamma_0^1(\mathfrak{N})_{\mathfrak{b}}$, $\Gamma_1^1(\mathfrak{N})_{\mathfrak{b}}$, and also $\Gamma_1(\mathfrak{N})_{\mathfrak{b}}$ if $\mathfrak{N}$ is a squarefree ideal. For a general congruence subgroup Γ such that $\mathrm{P}\Gamma(\mathfrak{N})_{\mathfrak{b}} \le \mathrm{P}\Gamma$, a promising computational approach is to resolve the cusps for the action of $\Gamma(\mathfrak{N})_{\mathfrak{b}}$ and quotient the resulting smooth surface by the finite group $\mathrm{P}\Gamma_{\mathfrak{b}}/\,\mathrm{P}\Gamma(\mathfrak{N})_{\mathfrak{b}}$; we do not pursue this further in this paper.

Once $G(M,V)$ is known, the intersection numbers we are looking for can be explicitly computed as follows. Let $v, v' : F \hookrightarrow \mathbb{R}$ be the two embeddings of F in $\mathbb{R}$. We say that a $\mathbb{Z}$-basis (x,y) of M is oriented if

$$v(x)v'(y) - v'(x)v(y) > 0. \tag{3.3.2}$$

Necessarily, exactly one of (x,y) or (y,x) is oriented. Given an oriented basis (x,y) of M, we compute the Hirzebruch–Jung continued fraction of $v(x/y)$, as follows:

$$v(x/y) = b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \dots}} =: [[b_0, b_1, \dots]], \tag{3.3.3}$$

where b_0 is the smallest integer greater than $v(x/y)$, then b_1 is the smallest integer greater than $1/(v(x/y) - b_0)$, and so forth. This continued fraction is periodic; let d be its period. For each $1 \le j \le d$, we define

$$w_j := [[b_j, b_{j+1}, \dots]].$$

Let $k \ge 1$ be minimal such that $(w_1 \cdots w_d)^k \in V$. Then the resolution of the cusp $(a:c)$ is a cyclic configuration of dk curves isomorphic to $\mathbb{P}^1$, with intersection numbers $(-b_0, \dots, -b_{d-1})$ repeated k times, except when $dk = 1$, in which case we find one rational curve with an ordinary double point and self-intersection $-b_0 + 2$ [17, Chapter II, Lemma 3.2].

Our new contribution in this section is the explicit description of the type of a cusp $(a:c)$ for standard congruence subgroups. Without loss of generality, we can assume $\mathfrak{b}$ and $\mathfrak{N}$ to be coprime. We also normalize our cusp representatives $(a:c)$ as follows: we assume that $a \in R$, $c \in \mathfrak{b}$, and that the (now integral) ideal $\mathfrak{s} = aR + c\mathfrak{b}^{-1}$ is coprime to $\mathfrak{N}$.

Lemma 3.3.4. *Let $\mathfrak{s} = aR + c\mathfrak{b}^{-1}$. Then there exists $\lambda \in \mathfrak{s}^{-1}$ and $\mu \in \mathfrak{s}^{-1}\mathfrak{b}^{-1}$ such that*

$$\gamma = \begin{pmatrix} \lambda & \mu \\ -c & a \end{pmatrix} \in \begin{pmatrix} \mathfrak{s}^{-1} & \mathfrak{s}^{-1}\mathfrak{b}^{-1} \\ \mathfrak{s}\mathfrak{b} & \mathfrak{s} \end{pmatrix} \cap \mathrm{GL}_2^+(F) \tag{3.3.5}$$

satisfies $\det(\gamma) = 1$ *and* $\gamma(a:c) = \infty$.

Proof. Following van der Geer [17, Proposition I.1.1], write

$$1 \in \mathfrak{s}^{-1}\mathfrak{s} = a\,\mathfrak{s}^{-1} + c\,\mathfrak{s}^{-1}\mathfrak{b}^{-1}. \tag{3.3.6}$$

Therefore there exist $\lambda \in \mathfrak{s}^{-1}$ and $\mu \in \mathfrak{s}^{-1}\mathfrak{b}^{-1}$ such that $\lambda a + \mu c = 1$. ☐

We now describe the types $G(M,V)$ of each cusp for the level subgroups listed above.

Proposition 3.3.7. *Let $\mathfrak{s} = aR + c\mathfrak{b}^{-1}$ as above, and let $\mathfrak{M} = \mathfrak{M}(a,c) = \mathfrak{N} + c(\mathfrak{b}\mathfrak{s})^{-1}$. Then*

(a) *For level $\Gamma_0(\mathfrak{N})_{\mathfrak{b}}$, the cusp $(a:c)$ is of type $G(M,V)$ where*

$$M = \mathfrak{s}^{-2}\mathfrak{b}^{-1}\mathfrak{N}\,(\mathfrak{N} + \mathfrak{M}^2)^{-1}, \text{ and}$$
$$V = \left\{ v \in R_{>0}^{\times} : v \equiv 1 \pmod{\mathfrak{M} + \mathfrak{M}/\mathfrak{N}} \right\}.$$

(b) *For level* $\Gamma_0^1(\mathfrak{N})_{\mathfrak{b}}$*, the cusp* $(a:c)$ *is of type* $G(M,V)$ *where*

$$\begin{aligned} &M = \mathfrak{s}^{-2}\mathfrak{b}^{-1}\mathfrak{N}\,(\mathfrak{N}+\mathfrak{M}^2)^{-1}, \ \textit{and} \\ &V = \bigl\{v^2 : v \in R^\times \text{ and } v^2 \equiv 1 \pmod{\mathfrak{N}+\mathfrak{N}/\mathfrak{M}}\bigr\}. \end{aligned}$$

(c) *For level* $\Gamma_1^1(\mathfrak{N})_{\mathfrak{b}}$*, the cusp* $(a:c)$ *is of type* $G(M,V)$ *where*

$$\begin{aligned} &M = \mathfrak{s}^{-2}\mathfrak{b}^{-1}(\mathfrak{N}/\mathfrak{M}), \ \textit{and} \\ &V = \bigl\{v^2 : v \in R^\times \text{ and } v \equiv 1 \pmod{\mathfrak{N}\mathfrak{M}\,(\mathfrak{N}+\mathfrak{M}^2)^{-1}}\bigr\}. \end{aligned}$$

(d) *Assume that* $\mathfrak{N}$ *is squarefree. Let* $U \subset R^\times$ *be the subgroup of units congruent to* 1 *modulo* $\mathfrak{N}/\mathfrak{M}$*. Then for level* $\Gamma_1(\mathfrak{N})_{\mathfrak{b}}$*, the cusp* $(a:c)$ *is of type* $G(M,V)$ *where*

$$\begin{aligned} &M = \mathfrak{s}^{-2}\mathfrak{b}^{-1}(\mathfrak{N}/\mathfrak{M}), \ \textit{and} \\ &V = \bigl\{v \in R^\times_{>0} : v \equiv \varepsilon \pmod{\mathfrak{M}} \text{ for some } \varepsilon \in U\bigr\}. \end{aligned}$$

In each case, we can explicitly compute a matrix $\gamma \in \mathrm{GL}_2^+(F)$ *sending* $(a:c)$ *to* ∞ *such that*

$$\gamma^{-1}\begin{pmatrix} v & m \\ 0 & 1 \end{pmatrix}\gamma \tag{3.3.8}$$

lies in the chosen congruence subgroup of $\Gamma_{\mathfrak{b}}$ *(as a transformation, i.e. up to scalars) precisely when* $(v,m) \in V \times M$*.*

Proof. Let $\gamma = \left(\begin{smallmatrix} \lambda & \mu \\ -c & a \end{smallmatrix}\right)$ be as in Lemma 3.3.4. As transformations of $\mathcal{H}$, elements in the stabilizer of $(a:c)$ take the form

$$N = \gamma^{-1}\begin{pmatrix} v & m \\ 0 & 1 \end{pmatrix}\gamma = \begin{pmatrix} 1 + a\bigl(\lambda(v-1) - cm\bigr) & a\bigl(\mu(v-1) + am\bigr) \\ c\bigl(\lambda(v-1) - cm\bigr) & v - a\bigl(\lambda(v-1) - cm\bigr) \end{pmatrix} \tag{3.3.9}$$

for some $v \in F^\times_{>0}$ and $m \in F$. Then N lies in $\Gamma(1)_{\mathfrak{b}}$ if and only if $v \in R^\times_{>0}$ and $m \in \mathfrak{s}^{-2}\mathfrak{b}^{-1}$. Note that by multiplying γ on the left by a matrix of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \tag{3.3.10}$$

we can make an affine change of variables of the form $m \mapsto m + x(v-1)$ for any $x \in F$, provided that $x(v-1)$ remains in $s^{-2}\mathfrak{b}^{-1}$.

Factor $\mathfrak{N} = \prod_{i\in I}\mathfrak{p}_i^{e_i}$, where each $\mathfrak{p}_i$ is a prime ideal of R, and write $\mathfrak{M} = \prod_{i\in I}\mathfrak{p}_i^{f_i}$, where $0 \le f_i \le e_i$ for all i. The exponents f_i are related to the factorization of c: we can write

$$c = c'\prod_{i\in I}\mathfrak{p}_i^{f_i} \tag{3.3.11}$$

where c' is prime to $\mathfrak{p}_i$ whenever $f_i < e_i$. We now separate the four cases.

(a) *Level* $\Gamma_0(\mathfrak{N})_{\mathfrak{b}}$. We have $N \in \Gamma_0(\mathfrak{N})_{\mathfrak{b}}$ if and only if for all $i \in I$, we have

$$\lambda(v-1) - cm \equiv 0 \pmod{\mathfrak{p}_i^{e_i - f_i}}. \tag{3.3.12}$$

If c is invertible modulo $\mathfrak{p}_i$, then after an affine change of variables as above, this relation becomes $m \equiv 0 \pmod{\mathfrak{p}_i^{e_i}}$. Otherwise, λ is invertible modulo $\mathfrak{p}_i$ (because $\det g$ is a unit, and $\mathfrak{s}$ and $\mathfrak{b}$ are coprime to $\mathfrak{N}$). A necessary condition is that $v \equiv 1 \pmod{\mathfrak{p}_i^{\min\{f_i, e_i - f_i\}}}$. Then another affine change of variables brings this relation into the form $m \equiv 0 \pmod{\mathfrak{p}_i^{\max\{e_i - 2f_i, 0\}}}$.

(b) *Level* $\Gamma_0^1(\mathfrak{N})_{\mathfrak{b}}$. Compared to the case of $\Gamma_0(\mathfrak{N})_{\mathfrak{b}}$, we only have to add the condition that v is a square.

(c) *Level* $\Gamma_1^1(\mathfrak{N})_{\mathfrak{b}}$. We have $N \in \Gamma_1^1(\mathfrak{N})_{\mathfrak{b}}$ up to multiplication by a scalar matrix if and only if the following conditions hold for all $i \in I$:

- $v = w^2$ is a square,
- $\lambda(w - w^{-1}) - cm \equiv 0 \pmod{\mathfrak{p}_i^{e_i - f_i}}$, and
- $w^{-1} + a\big(\lambda(w - w^{-1}) - cm\big) = w + c\big(\mu(w^{-1} - w) - am\big)$ is equal to 1 modulo $\mathfrak{p}_i^{e_i}$.

This implies a third relation

$$w - a\big(\lambda(w - w^{-1}) - cm\big) \equiv 1 \pmod{\mathfrak{p}_i^{e_i}} \tag{3.3.13}$$

which can also be deduced from the determinant 1 condition. If $f_i = 0$, we can rewrite this system as $m \equiv 0 \pmod{\mathfrak{p}_i^{e_i}}$ and $w \equiv 1 \pmod{\mathfrak{p}_i^{e_i}}$. Otherwise, we obtain $w \equiv 1 \pmod{\mathfrak{p}_i^{f_i}}$ from the second equation. Write $w = 1 + \eta$ where the valuation of η at $\mathfrak{p}_i$ is at least f_i. Summing the second and third relations, we see that the valuation of η must be at least $e_i/2$. The equations become:

$$2\lambda\eta - cm \equiv 0 \pmod{\mathfrak{p}_i^{e_i - f_i}}, \quad a\big(2\lambda\eta - cm\big) \equiv \eta \pmod{\mathfrak{p}_i^{e_i}}. \tag{3.3.14}$$

Since a is invertible modulo $\mathfrak{p}_i$, the valuation of 2η must be at least $e_i - f_i$. (Note that $\max\{e_i - f_i, f_i\} \geq e_i/2$ always.) The final equation can also be rewritten as $(1 + 2\mu c)\eta = acm \pmod{\mathfrak{p}_i^{e_i}}$. Since the valuation of η is at least f_i, an affine change of variables brings this relation into the form $m = 0 \pmod{\mathfrak{p}_i^{e_i - f_i}}$.

(d) *Level* $\Gamma_1(\mathfrak{N})_{\mathfrak{b}}$. In general, we have $N \in \Gamma_1(\mathfrak{N})_{\mathfrak{b}}$ up to multiplication by a scalar matrix if and only if there exists a unit $\varepsilon \in R^\times$ such that the following conditions hold for all $i \in I$:

$$\begin{gathered} \lambda(v - 1) - cm \equiv 0 \pmod{\mathfrak{p}_i^{e_i - f_i}}, \text{ and} \\ 1 + a\big(\lambda(v - 1) - cm\big) = v + c\big(\mu(1 - v) - am\big) \equiv \varepsilon \pmod{\mathfrak{p}_i^{e_i}}. \end{gathered} \tag{3.3.15}$$

Assuming $\mathfrak{N}$ squarefree, the only cases are $f_i = 0$ and $f_i = 1$ (both with $e_i = 1$). In the first case, after a change of variables, the system becomes $m \equiv 0 \pmod{\mathfrak{p}_i}$ and $\varepsilon \equiv 1 \pmod{\mathfrak{p}_i}$, thus $\varepsilon \in U$. In the second case, the system reads $v \equiv \varepsilon \pmod{\mathfrak{p}_i}$.

The required forms of $G(M, V)$ are then obtain from recombining the local conditions. In each case, a suitable global change of variables $m \mapsto m + x(v - 1)$ can be computed using the Sun Zi theorem (CRT) to obtain the desired matrix γ. ☐

4. ELLIPTIC POINTS

In general, a congruence subgroup will not act freely on $\mathcal{H}$. In this section, we seek to understand algorithmically the fixed points of the action. The key idea will be to relate the number of elliptic points with optimal embedding numbers of quadratic orders [18, Chapters 30 and 39].

4.1. **Setup.** Let $\Gamma \le \mathrm{GL}_2^+(F)$ be a congruence subgroup and let $\mathcal{O}$ be the R-order in $\mathrm{M}_2(F)$ generated by Γ. By [17, §I.5], if $\gamma \in \Gamma \smallsetminus R^\times$ has a fixed point $z \in \mathcal{H}$, then $\mathrm{tr}(\gamma)^2 - 4\det(\gamma)$ is totally negative. In this case we call γ elliptic and its fixed point an elliptic point for Γ. For any $z \in \mathcal{H}$, the stabilizer $\mathrm{Stab}_\Gamma(z) := \{\gamma \in \Gamma : \gamma z = z\}$ contains $R^\times \cap \Gamma$ (acting trivially on $\mathcal{H}$) and the quotient $\mathrm{Stab}_\Gamma(z)/(R^\times \cap \Gamma)$ is a finite cyclic group. When this cyclic group is nontrivial, we obtain a quadratic CM (totally imaginary) extension $K := F(\gamma)$ and an order $S := R[\gamma] \subseteq K$, where γ is a generator of $\mathrm{Stab}_\Gamma(z)/(R^\times \cap \Gamma)$. The elliptic condition ensures that only finitely such quadratic orders arise up to isomorphism. Conversely, if $\phi\colon R[\gamma] \hookrightarrow \mathcal{O}$ is an embedding, then γ has a unique isolated fixed point z in $\mathcal{H}$. If ϕ' is an embedding conjugate to ϕ under Γ, then $\phi(\gamma)$ and $\phi'(\gamma)$ fix the same elliptic point in $\mathcal{H}$. Because γ has finite order, $\overline{\gamma} \in \langle\gamma\rangle$, so we may always compose ϕ by the involution of $\mathcal{O}$ to obtain a new morphism $\overline{\phi}\colon R[\gamma] \to \mathcal{O}$ with identical image, i.e., associated to the same elliptic point. We do not count ϕ and $\overline{\phi}$ separately. If K/F be a CM-extension and S is an R-order, then $S^\times/R^\times$ is cyclic. In particular, all elliptic elements of S are contained in the same cyclic subgroup.

Consequently, there is a bijection between conjugacy classes of pairs $\phi, \overline{\phi}\colon S \hookrightarrow \mathcal{O}$ of embeddings and elliptic points. Thus, one can compute elliptic points in two steps. First, determine the list of R-orders $R[\gamma]$ such that γ generates a quadratic CM extension. Second, for each S in the aforementioned list, determine the number of embeddings of S into $\mathcal{O}$ up to Γ-conjugacy. The first task is taken care of by the following lemma.

Lemma 4.1.1. *Let $\gamma \in \Gamma$ be a nontrivial torsion element.*

(a) *If $\Gamma = \mathrm{PSL}_2(R)$, then $R[\gamma] \cong R[\zeta]$ for some root of unity ζ such that $\zeta + \zeta^{-1} \in F$. In particular, when F is a real quadratic field we have $\zeta \in \{\zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_8, \zeta_{10}\}$.*

(b) *If $\Gamma = \mathrm{PGL}_2^+(R)$ and γ has order m, then $m \in \{2, 3, 4, 5, 6, 12\}$, and $R[\gamma]$ is isomorphic to an order of the form $S := R[x]/(f(x))$, where $f(x)$ is an irreducible quadratic factor of $x^m \pm \alpha$ for some totally positive unit α. Furthermore, the isomorphism class of S is independent of the class of α in $R^{\times m}$.*

The collection of minimal polynomials from Lemma 4.1.1 is effectively computable. We let Ω_q denote the isomorphism classes of R-orders S such that $S \cong R[\gamma]$ and $\langle\gamma\rangle = S^\times/R^\times$ has order q.

We now turn to counting embeddings of a given order. Let K be a quadratic CM extension of F with ring of integers $\mathbb{Z}_K$, and let $S \subset K$ be an order of K. We also introduce the following notation: $\mathcal{O}^1$ and $\mathcal{O}^\times_{>0}$ denote the subgroups of $\mathcal{O}^\times$ consisting of matrices γ such that $\det(\gamma) = 1$ or $\det(\gamma) \in F^\times_{>0}$, respectively.

We say that an embedding $\phi\colon S \hookrightarrow \mathcal{O}$ is optimal if $\phi(K) \cap \mathcal{O} = \phi(S)$. Not all embeddings of S are necessarily optimal; however, given an embedding ϕ, there exists a unique order $S \subseteq S' \subset K$ such that S' is optimally embedded under ϕ, namely $S' := \phi^{-1}(\phi(K) \cap \mathcal{O})$. We then obtain the decomposition

$$\{\text{embeddings of } S \text{ into } \mathcal{O}\} = \bigsqcup_{S \subseteq S' \subset K} \{\text{optimal embeddings } S' \hookrightarrow \mathcal{O}\}. \tag{4.1.2}$$

It is well-known how to enumerate such superorders S'. Let $\mathfrak{f}_S$ be the conductor of S. Then for every divisor $\mathfrak{d} \mid \mathfrak{f}_S$, there exists an order $S' \subset K$ containing S with conductor $\mathfrak{d}$ (as a $\mathbb{Z}_K$-module, so $\mathbb{Z}_K$ has trivial conductor).

For a group $\mathcal{O}^1 \subset \Gamma \subset \mathcal{O}^\times$, we denote by $m(S, \mathcal{O}; \Gamma)$ the number of optimal embeddings of S into $\mathcal{O}$ up to Γ-conjugacy. The previous decomposition shows that

$$\#\{\Gamma\text{-conjugacy classes of } \phi\colon S \hookrightarrow \mathcal{O}\} = \sum_{S \subseteq S' \subset K} m(S', \mathcal{O}; \Gamma). \tag{4.1.3}$$

The property of being an optimal embedding is local, so the numbers $m(S', \mathcal{O}; \Gamma)$ can be computed adelically. The field K does not satisfy the selectivity condition (OS) [18, 31.1.6, condition (a) of Proposition 31.2.1] since $B = \mathrm{M}_2(F)$ is split at all real places. Thus for every R-order $S' \subseteq K$ we have

$$m(S', \mathcal{O}; \mathcal{O}^\times) = \frac{h(S)}{h(R)} m(\widehat{S}', \widehat{\mathcal{O}}; \widehat{\mathcal{O}}^\times), \tag{4.1.4}$$

where $h(S) = \#\operatorname{Pic} S$ [18, Corollary 31.1.10]. Here, $m(\widehat{S}', \widehat{\mathcal{O}}; \widehat{\mathcal{O}}^\times)$ denotes the number of adelically optimal embeddings of the adelic order $\widehat{S}'$, up to $\widehat{\mathcal{O}}$-conjugacy. To recover $m(S', \mathcal{O}; \Gamma)$, we use the formula [18, Lemma 30.3.14]

$$m(S', \mathcal{O}; \Gamma) = m(S', \mathcal{O}; \mathcal{O}^\times)[\operatorname{nrd}(\mathcal{O}^\times) : \operatorname{nrd}(\Gamma) \operatorname{nrd}(S'^\times)]. \tag{4.1.5}$$

The adelic embedding numbers $m(\widehat{S}, \widehat{\mathcal{O}}; \widehat{\mathcal{O}}^\times)$, a product of (finitely many) local embedding numbers, are given explicitly in [18, §30.6 and §30.7].

4.2. **Formula.** For simplicity, we concentrate on the case of the modular groups $\Gamma_0(\mathfrak{N})_\mathfrak{b}$ and $\Gamma_0^1(\mathfrak{N})_\mathfrak{b}$. By (2.1.6), we may and do suppose that $\mathfrak{b}$ is coprime to the order of any torsion point, i.e., coprime to $2, 3, 5$. Consider the order

$$\mathcal{O} = \mathcal{O}_0(\mathfrak{N})_\mathfrak{b} = \begin{pmatrix} R & \mathfrak{b}^{-1} \\ \mathfrak{N}\mathfrak{b} & R \end{pmatrix} \tag{4.2.1}$$

and notice that $\mathcal{O}^1 = \Gamma_0^1(\mathfrak{N})_\mathfrak{b}$ and $\mathcal{O}^\times_{>0} = \Gamma_0(\mathfrak{N})_\mathfrak{b}$.

For $q \geq 2$, we denote by m_q^1 and m_q^+ the number of elliptic points of order q in $\mathcal{O}^1\backslash\mathcal{H}$ and $\mathcal{O}^\times_{>0}\backslash\mathcal{H}$, respectively. From the previous section we have that

$$\begin{aligned} m_q^1 &= \frac{1}{2} \sum_{\substack{S \supseteq R[\zeta_{2q}] \\ \#S^\times_{\mathrm{tors}} = 2q}} m(S, \mathcal{O}; \mathcal{O}^1), \\ m_q^+ &= \frac{1}{2} \sum_{S \in \Omega_q} \sum_{\substack{S' \supset S \\ \#S'^\times / R^\times = q}} m(S', \mathcal{O}; \mathcal{O}^\times_{>0}). \end{aligned} \tag{4.2.2}$$

Proposition 4.2.3. *We have*

$$m_q^1 = \frac{2^{n-1}}{h(R)} \sum_S \frac{h(S)}{Q(S)} m(\widehat{S}, \widehat{\mathcal{O}}; \widehat{\mathcal{O}}^\times) \tag{4.2.4}$$

and

$$m_q^+ = \frac{2^{n-1}}{h^+(R)} \sum_S h(S)\, m(\widehat{S}, \widehat{\mathcal{O}}; \widehat{\mathcal{O}}^\times), \tag{4.2.5}$$

where S runs over orders as in (4.2.2) and $Q(S)$ is the Hasse unit index.

Proof. To compute m_q^1 we repeat the argument in Voight [18, Proposition 39.4.12]. The only difference is that $\left[R_{>_\Omega 0}^\times : R^{\times 2}\right] = [R^\times : R^{\times 2}] = 2^n$ because $\Omega = \emptyset$, i.e., there are no real ramified places in B. For m_q^+, we repeat the same calculation with $\mathcal{O}_{>0}^\times$, so

$$\left[\operatorname{nrd}\left(\mathcal{O}_{>0}^\times\right)\operatorname{nrd}\left(S^\times\right) : R^{\times 2}\right] = \left[R_{>0}^\times : R^{\times 2}\right] = h^+(R)/h(R). \tag{4.2.6}$$ ☐

Note that the number of elliptic points of a given order q is independent of the component $\mathfrak{b}$. However, as we discuss in the next section, the rotation types may differ; consequently, their contributions to surface invariants are not necessarily the same.

4.3. **Rotation factors.** We follow Prestel [14]. Let $z = (z_v)_v \in \mathcal{H}$ be an elliptic fixed point with stabilizer group $\langle\gamma\rangle \leq \Gamma_0(\mathfrak{N})_\mathfrak{b}$; let $\operatorname{tr}(\gamma) = t$ and $\det(\gamma) = u$, so that $\gamma^2 - t\gamma + u = 0$ and $t^2 - 4u$ is totally negative. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then

$$z_v = \frac{a_v - d_v}{2c_v} + \frac{1}{2|c_v|}\sqrt{t_v^2 - 4u_v} \tag{4.3.1}$$

(where we choose roots in the upper half-plane). The transformation $z' \mapsto (z' - z)/(z' - \overline{z})$ maps z to $(0, \dots, 0)$; the elliptic matrix γ then acts as a rotation $z \mapsto \zeta z = (\zeta_v z_v)_v$ of the product of n unit discs, rotating the v-component by

$$\zeta_v = \left(\frac{t_v^2}{2u_v} - 1\right) - i \operatorname{sgn}(t_v c_v)\sqrt{1 - \left(\frac{t_v^2}{2u_v} - 1\right)^2}, \tag{4.3.2}$$

so each ζ_v is a primitive root of unity whose order matches γ. We call the tuple $\zeta = (\zeta_v)_v$ the rotation factor of z.

If z' is another fixed point with stabilizer $\langle\gamma'\rangle$ with again $\operatorname{tr}(\gamma') = t$ and $\det(\gamma') = u$, then there exists $\beta \in \mathrm{GL}_2(F)$ such that $\gamma' = \beta^{-1}\gamma\beta$ (by the Skolem–Noether theorem). If γ has rotation factor ζ, then γ' has rotation factor

$$\zeta' = \zeta^{\operatorname{sgn}(\det(\beta))}; \tag{4.3.3}$$

in particular, they have the same rotation factor if and only if $\det(\beta) > 0$ is totally positive.

Connecting this up with the previous section, we need consider optimal embeddings but with attention to signs of the determinant—we will encounter a modified form of the selectivity phenomenon, so we follow the notation and conventions in Voight [18, Chapter 31]. Let $K = F(\gamma) \supseteq F$ be a CM field. Choose a fixed embedding $K \hookrightarrow \mathrm{M}_2(F)$ and identify K with its image; this will serve as a reference point (like a base point of a fundamental group, see also below).

Definition 4.3.4. An embedding $\phi_\beta \colon K \hookrightarrow \mathrm{M}_2(F)$ corresponding to conjugation by an element $\beta \in \mathrm{GL}_2(F)$ is oriented if $\det(\beta) > 0$ is totally positive.

Let $S \subset K$ be an R-order and let $\mathcal{O} \subset \mathrm{M}_2(F)$ be an R-order. An (optimal) embedding $\phi \colon S \hookrightarrow \mathcal{O}$ determines an embedding $K \hookrightarrow \mathrm{M}_2(F)$, so we similarly define such an order to be oriented. Let $\operatorname{Emb}^+(S, \mathcal{O})$ be the set of oriented optimal embeddings $\phi \colon S \hookrightarrow \mathcal{O}$ and let $\operatorname{Emb}^+(S, \mathcal{O}; \mathcal{O}_{>0}^\times)$ be this set up to conjugation by $\mathcal{O}_{>0}^\times$. Let

$$E^+ := \{\beta \in \mathrm{GL}_2^+(F) : \beta^{-1}K\beta \cap \mathcal{O} = \beta^{-1}S\beta\}. \tag{4.3.5}$$

Then [18, (30.3.13)] the map which sends the class of β to conjugation by β induces a bijection

$$K^\times\backslash E^+/\mathcal{O}_{>0}^\times \xrightarrow{\sim} \operatorname{Emb}^+(S,\mathcal{O};\mathcal{O}_{>0}^\times). \tag{4.3.6}$$

Let

$$m^+(S,\mathcal{O};\mathcal{O}_{>0}^\times) := \#\operatorname{Emb}^+(S,\mathcal{O};\mathcal{O}_{>0}^\times). \tag{4.3.7}$$

With this notation in hand, we can generalize the notion of selectivity to oriented optimal embeddings as follows. Let $\operatorname{Gen}\mathcal{O}$ be the genus of the order $\mathcal{O}$, i.e. the set of orders in $\mathrm{M}_2(F)$ which are locally isomorphic to $\mathcal{O}$.

Definition 4.3.8. We say that $\operatorname{Gen}\mathcal{O}$ is orientedly genial for S if $\operatorname{Emb}^+(S,\mathcal{O}') \neq \emptyset$ for all $\mathcal{O}' \in \operatorname{Gen}\mathcal{O}$; otherwise, we say that $\operatorname{Gen}\mathcal{O}$ is orientedly optimally selective.

In other words, $\operatorname{Gen}\mathcal{O}$ is orientedly genial for S if and only if S has an oriented embedding into every order $\mathcal{O}'$ that is locally isomorphic to $\mathcal{O}$. In terms of rotation factors, if γ has rotation factor ζ and $S \supseteq R[\gamma]$, then $\operatorname{Gen}\mathcal{O}$ is orientedly genial for S if and only if every order $\mathcal{O}'$ locally isomorphic to $\mathcal{O}$ admits an optimal embedding of S with rotation factor ζ.

We attach the group [18, (28.5.8)]

$$GN^+(\mathcal{O}) := F_{>0}^\times \det(N_{\mathrm{GL}_2(\widehat{F})}(\widehat{\mathcal{O}})) \leq \widehat{F}^\times. \tag{4.3.9}$$

to the order $\mathcal{O}$, as well as the class group

$$\operatorname{Cl}_{GN^+(\mathcal{O})} R := \widehat{F}^\times/GN^+(\mathcal{O}). \tag{4.3.10}$$

The determinant map induces a bijection between $\operatorname{Typ}^+\mathcal{O}$, the similarly defined oriented type set of $\mathcal{O}$, and $\operatorname{Cl}_{GN^+(\mathcal{O})} R$ [18, Corollary 28.5.10]. By class field theory, attached to $GN^+(\mathcal{O})$ is an abelian extension $H_{GN^+(\mathcal{O})} \supseteq F$.

We define the following condition, called the orientedly optimal selectivity condition:

(OOS) K is a subfield of the class field $H_{GN^+(\mathcal{O})} \supseteq F$.

We now restrict to the case at hand: we suppose that $\mathcal{O}$ is an Eichler order of level $\mathfrak{N}$ [18, §31.2]. Then there is a surjection $\operatorname{Cl}^+ R \to \operatorname{Cl}_{GN^+(\mathcal{O})} R$, presenting this class group as the quotient of the narrow class group by the primes $\mathfrak{p} \mid \mathfrak{N}$ with $\operatorname{ord}_{\mathfrak{p}}(\mathfrak{N})$ odd. Therefore (OOS) holds if and only if the following two conditions hold:

(i) K is unramified at all nonarchimedean places $v \in \operatorname{Pl} F$; and
(ii) If $\mathfrak{p} \mid \mathfrak{N}$ with $\operatorname{ord}_{\mathfrak{p}}\mathfrak{N}$ odd, then $\mathfrak{p}$ splits in K.

Theorem 4.3.11 (Oriented optimal selectivity)**.** *Suppose that $\mathcal{O}$ is an Eichler order. Then the following statements hold.*

(a) $\operatorname{Gen}\mathcal{O}$ *is orientedly optimally selective for S if and only if the orientedly optimal selectivity condition (OOS) holds.*
(b) *If* $\operatorname{Gen}\mathcal{O}$ *is orientedly optimally selective for S, then* $\operatorname{Emb}^+(S,\mathcal{O}') \neq \emptyset$ *for precisely* half *of the types* $[\mathcal{O}'] \in \operatorname{Typ}^+\mathcal{O}$. *More precisely, if* $[\mathcal{O}'] \leftrightarrow [\mathfrak{b}] \in \operatorname{Cl}^+ R$, *then* $\operatorname{Emb}^+(S,\mathcal{O}') \neq \emptyset$ *if and only if* $\operatorname{Frob}_{\mathfrak{b}} \in \operatorname{Gal}(K\,|\,F)$ *is trivial.*
(c) *In all cases,*

$$m(S,\mathcal{O}';\mathcal{O}'^\times_{>0}) = m(S,\mathcal{O};\mathcal{O}_{>0}^\times) \tag{4.3.12}$$

for all $\mathcal{O}' \in \operatorname{Gen}\mathcal{O}$ *whenever both sides are nonzero.*

Proof. Amazingly, the proof given by Voight [18, Main Theorem 31.1.7] applies, *mutatis mutandis* starting with (4.3.6). The selectivity sandwich [18, 31.3.14], once oriented, reads

$$F_{>0}^{\times}\det(\widehat{K}^{\times}) \overset{\text{(OOS)}}{\leq} \det(\widehat{K}^{\times})GN^{+}(\mathcal{O}) \overset{m}{\leq} F_{>0}^{\times}\det(\widehat{E}) \overset{s}{\leq} \widehat{F}^{\times}. \tag{4.3.13}$$

Again, the first inequality (OOS) is an equality if and only if the orientedly optimal selective condition (OOS) holds, the middle inequality m is always an equality (implying equality of oriented embedding numbers when nonzero), and the final inequality s is an equality if and only if there is no oriented selectivity obstruction. The refinement in part (b) is also proven in the same way [18, Proposition 31.4.4, Corollary 31.4.6]. ☐

Corollary 4.3.14. *Suppose that $K = F(\gamma) \supseteq F$ is ramified at a nonarchimedean place $v \in \operatorname{Pl} F$. Then all rotation factors for γ occur equally across the groups $\Gamma_0(\mathfrak{N})_{\mathfrak{b}}$.*

Proof. Apply Theorem 4.3.11, noting that condition (i) above fails. ☐

We now return to the issue of the choice of reference embedding $K \hookrightarrow \mathrm{M}_2(F)$.

Lemma 4.3.15. *Assume that $\operatorname{Gen}\mathcal{O}$ is orientedly optimally selective for S. Let $S \subseteq S_0$ be a suborder of conductor $\mathfrak{f} = \mathfrak{f}(S_0|S)$, and assume that $\operatorname{Emb}^{+}(S_0,\mathcal{O}) \neq \emptyset$. Then $\operatorname{Emb}^{+}(S,\mathcal{O})$ is nonempty if and only if $\operatorname{Frob}_{\mathfrak{f}} \in \operatorname{Gal}(K \,|\, F)$ is trivial.*

Proof. Let $\widehat{\phi}\colon \widehat{S}_0 \hookrightarrow \widehat{\mathcal{O}}$ be a local orientedly optimal embedding, corresponding to $\widehat{\beta} \in \mathrm{GL}_2(\widehat{F})$ with the property that $\widehat{\beta}^{-1}\widehat{K}\widehat{\beta} \cap \widehat{\mathcal{O}} = \widehat{\phi}(\widehat{S}_0)$. Then $\det(\widehat{\beta}) = \widehat{f}$ where $\widehat{f}\widehat{R} \cap F = \mathfrak{f}$. Exactly as in the proof of Theorem 4.3.11(b), $\widehat{\beta}$ yields an orientedly optimal embedding $\phi\colon S_0 \hookrightarrow \mathcal{O}$ if and only if $\widehat{f} \in F_{>0}^{\times}\det(\widehat{K}^{\times})$ if and only if $\operatorname{Frob}_{\mathfrak{f}} \in \operatorname{Gal}(K \,|\, F)$ is trivial. ☐

Corollary 4.3.16. *Let K be a CM field with $S \subset K$ an R-order, and let $\gamma \in S^{\times} \smallsetminus R^{\times}$ be such that $\gamma R^{\times} \in S^{\times}/R^{\times}$ has finite order and generates $(S^{\times}/R^{\times})_{\text{tors}}$. Let $f_\gamma(x) = x^2 - t_\gamma x + u_\gamma \in R[x]$ be the minimal polynomial of γ. Suppose that the condition (OOS) holds. Then the following statements hold.*

(a) *The order $R[\gamma]$ embeds orientedly optimally into*

$$\mathcal{O}_0(\mathfrak{N})_{(1)} = \begin{pmatrix} R & R \\ \mathfrak{N} & R \end{pmatrix} \tag{4.3.17}$$

if and only if $\widehat{R}[\gamma] \hookrightarrow \widehat{\mathcal{O}}$ embeds optimally (i.e., if and only if $m(\widehat{R}[\gamma],\widehat{\mathcal{O}};\widehat{\mathcal{O}}^{\times}) \neq 0$) if and only if there exists $x \in R$ such that $f_\gamma(x) \equiv 0 \pmod{\mathfrak{N}}$, via the embedding

$$\phi(x) = \begin{pmatrix} x & 1 \\ -f_\gamma(x) & t_\gamma - x \end{pmatrix}. \tag{4.3.18}$$

(b) *Let $x \in R$ be such that $f_\gamma(x) \equiv 0 \pmod{\mathfrak{N}}$, and let ζ be the rotation type for (4.3.18). Let $\mathfrak{f} = \mathfrak{f}(S \,|\, R[\gamma])$ be the conductor of $R[\gamma] \subseteq S$, so $\operatorname{disc} R[\gamma] = \mathfrak{f}^2 \operatorname{disc} S$. Then the rotation factors $\zeta' = (\zeta_v^{\varepsilon_v})_v$ which occur for fixed points of optimal embeddings of S (with stabilizer of order q) into $\mathcal{O}_0(\mathfrak{N})_{\mathfrak{b}}$ are exactly those with*

$$\prod_v \varepsilon_v = \left(\frac{K}{\mathfrak{f}\mathfrak{b}}\right), \tag{4.3.19}$$

where $\left(\dfrac{K}{\mathfrak{f}\mathfrak{b}}\right) \in \{\pm 1\}$ is trivial if and only if $\operatorname{Frob}_{\mathfrak{f}\mathfrak{b}} \in \operatorname{Gal}(K \,|\, F)$ is trivial.

Proof. For (a), we choose the given rational canonical form $\phi\colon F(\gamma) \to M_2(F)$ as our reference point. (For more on the local statement and normalized embeddings, we refer to Voight [18, §30.6].)

For (b), starting with the reference point (a), we just combine Theorem 4.3.11(b) and Lemma 4.3.15 with the relationship to rotation factors given in (4.3.3): S embeds orientedly optimally into $\mathcal{O}_0(\mathfrak{N})_{\mathfrak{b}}$ if and only if $\mathrm{Frob}_{\mathfrak{f}\mathfrak{b}}$ is trivial in $\mathrm{Gal}(K \mid F)$. Considering now all possible orientations, we obtain exactly those with $\varepsilon = (\varepsilon_v)_v$ in the kernel of the composition of group homomorphisms

$$\{\pm 1\}^n \to \frac{\mathrm{Cl}^+ R}{\mathrm{Cl}\, R} \to \mathrm{Gal}(K \mid F) \tag{4.3.20}$$

giving the reformulation in (b). □

Remark 4.3.21. The results above generalize fully to the setup considered in Voight [18, Chapter 31], allowing a quaternion algebra B in place of $\mathrm{M}_2(F)$ over a global field F.

4.4. **Resolution of singularities.** Now suppose $n = 2$. Then van der Geer [17, §II.6] explains how to resolve the singularities at the elliptic points, as cyclic quotient singularities.

The resolution of an elliptic point of order q and rotation factor (ζ_q, ζ_q^r) is given by constructing a (finite) Hirzebruch–Jung continued fraction expansion $[[b_1, \dots b_d]]$ for q/r, as in section 3.3. Then there are d curves $C_1, \dots, C_d$ in the resolution chain, with self intersection numbers $C_1^2 = -b_1, \dots, C_d^2 = -b_d$. In addition $C_i \cdot C_j = \delta_{|i-j|,1}$ for $i \neq j$. The local Chern class of this chain is

$$\sum_{i=1}^{d} (x_i + y_i - 1) C_i, \tag{4.4.1}$$

the points $P_i = (x_i, y_i)$ being determined by $P_0 = (1, 0)$, $P_1 = \left(\frac{r}{q}, \frac{1}{q}\right)$ and $P_{i+1} = b_i P_i - P_{i-1}$.

5. DIMENSION FORMULAS

Another basic invariant of a Hilbert modular surface is the dimensions of spaces of cusp forms attached to it.

5.1. **Hilbert series.** The graded $M(\Gamma_0(\mathfrak{N}))$-module

$$S(\Gamma_0(\mathfrak{N})) := \bigoplus_{k \in 2\mathbb{Z}_{\geq 0}} S_k(\Gamma_0(\mathfrak{N})) \tag{5.1.1}$$

has a Hilbert series, defined by

$$\mathrm{Hilb}(S(\Gamma_0(\mathfrak{N}))) := \sum_{k \in 2\mathbb{Z}_{\geq 0}} \dim S_k(\Gamma_0(\mathfrak{N}))\, T^k \in \mathbb{Z}[[T]]. \tag{5.1.2}$$

This series gives insight into the algebraic structure of the ring of Hilbert modular forms for $\Gamma_0(\mathfrak{N})$, leading to possible further algorithmic explorations of Hilbert modular varieties. We remind the reader that this Hilbert series sums information from all connected components (section 2.3).

We compute the Hilbert series using an implementation of an explicit trace formula [2]. References for a trace formula for Hilbert modular forms include work of Saito [15, Theorem 2.1] for a general statement, and work of Okada [13, Theorem 2.1] for an explicit version when $\mathfrak{N} = (1)$. We recall our notation: F is a totally real field with degree n, ring of integers R, class number $h(R)$, and narrow class number $h^+(R)$. In this section, we assume that n is even. We denote by ζ_F the Dedekind zeta function of F, and by Nm the norm map for elements or ideals of F.

We consider pairs (u, t) where u ranges over $R^\times_{>0}/R^{\times 2}$, and for a fixed u, then $t \in R$ ranges over elements such that $t^2 - 4u$ is totally negative. For a given pair (u, t), we define the order $S(u, t) = R[x]/(x^2 - tx + u)$, and let $\mathfrak{f}(u, t)$ be its conductor, which is an ideal of R. Throughout, let $S \supseteq S(u, t)$ denote a superorder of $S(u, t)$ with class number $h(S)$.

From [2], we obtain the following expression of the Hilbert series:

$$\begin{aligned}\mathrm{Hilb}(S(\Gamma_0(\mathfrak{N}))) = A \cdot T^2 + B \cdot T\Big(T\frac{d}{dT}\Big)^n\Big(\frac{T}{1-T^2}\Big) \\ + \sum_{(u,t)} C(u,t) \sum_{m\geq 1} \mathrm{Nm}(D_{2m-2}(u,t))T^{2m},\end{aligned} \tag{5.1.3}$$

where

$$A := (-1)^{n-1} \cdot h^+(R),$$

$$B := \frac{1}{2^{n-1}} \cdot |\zeta_F(-1)| \cdot h(R) \cdot \mathrm{Nm}(\mathfrak{N}) \prod_{\mathfrak{p}\mid\mathfrak{N}} (1 + \mathrm{Nm}(\mathfrak{p})^{-1}),$$

$$C(u,t) := \frac{1}{2} \sum_{S\supseteq S(u,t)} \frac{h(S)}{[S^\times : R^\times]} m(\widehat{S}, \widehat{\mathcal{O}}; \widehat{\mathcal{O}}^\times), \tag{5.1.4}$$

$$\sum_{k\geq 0} D_k(u,t)T^k := \frac{1}{1 - tT + uT^2}.$$

The adelic embedding numbers $m(\widehat{S}, \widehat{\mathcal{O}}; \widehat{\mathcal{O}}^\times)$ were defined in section 4.1. In this context, we take $\mathcal{O}$ to be an Eichler order of level $\mathfrak{N}$ inside the definite quaternion algebra over F ramified at all infinite places.

The term $C(u, t)$ can be further described as follows. Let $K := F(x)/(x^2 - tx + u)$ be the CM extension of F containing $S(u, t)$. Denote its ring of integers by $\mathbb{Z}_K$, its unit group by $\mathbb{Z}_K^\times$, and its class number by $h(\mathbb{Z}_K)$. For an order $S \supseteq S(u, t)$ with conductor $\mathfrak{g}$ (so that $\mathfrak{g} \mid \mathfrak{f}(u, t)$) then

$$h(S) = \frac{h(\mathbb{Z}_K)}{[\mathbb{Z}_K^\times : S^\times]} \mathrm{Nm}(\mathfrak{g}) \prod_{\mathfrak{p}\mid\mathfrak{g}} \left(1 - \left(\frac{K}{\mathfrak{p}}\right) \mathrm{Nm}(\mathfrak{p})^{-1}\right). \tag{5.1.5}$$

Additionally, note that

$$m(\widehat{S}, \widehat{\mathcal{O}}; \widehat{\mathcal{O}}^\times) = \prod_{\mathfrak{p}\mid\mathfrak{N}} m_\mathfrak{p}(S, \mathcal{O}; \mathcal{O}^\times) \tag{5.1.6}$$

where $m_\mathfrak{p}(S, \mathcal{O}; \mathcal{O}^\times)$ is the number of local optimal embeddings of the order $S_\mathfrak{p}$ into $\mathcal{O}_\mathfrak{p}$ up to conjugation by $\mathcal{O}_\mathfrak{p}^\times$. These can be explicitly computed from [18, Proposition 30.6.12]. (See also see [18, Lemma 30.6.17] for the case when $\mathfrak{p}$ is an odd prime.) Hence

$$C(u,t) = \frac{h(\mathbb{Z}_K)}{2[\mathbb{Z}_K^\times : R^\times]} \sum_{\mathfrak{g}\mid\mathfrak{f}(u,t)} \mathrm{Nm}(\mathfrak{g}) \prod_{\mathfrak{p}\mid\mathfrak{g}} \left(1 - \left(\frac{K}{\mathfrak{p}}\right) \mathrm{Nm}(\mathfrak{p})^{-1}\right) \prod_{\mathfrak{p}\mid\mathfrak{N}} m_\mathfrak{p}(S(\mathfrak{g}), \mathcal{O}; \mathcal{O}^\times).$$

where $S(\mathfrak{g})$ is an order with conductor $\mathfrak{g}$.

5.2. **Explicit rational function.** We delve further to give an explicit expression for the Hilbert series as a rational function in T. For each (u, t), let $\alpha(u, t)$ and $\beta(u, t)$ be the roots of the polynomial $T^2 - tT + u$. Since the discriminant is totally negative (in particular nonzero), we can write

$$\frac{1}{1 - tT + uT^2} = \frac{1}{\alpha(u,t) - \beta(u,t)} \left(\frac{\alpha(u,t)}{1 - \alpha(u,t)T} - \frac{\beta(u,t)}{1 - \beta(u,t)T} \right). \tag{5.2.1}$$

Thus, for every $k \geq 0$,

$$D_k(u,t) = \frac{1}{\alpha(u,t) - \beta(u,t)} (\alpha(u,t)^{k+1} - \beta(u,t)^{k+1}). \tag{5.2.2}$$

Denote by L a Galois closure for $F/\mathbb{Q}$. Note that L is totally real, so the extension $L(\alpha(u,t))/L$ is of degree 2. The characteristic polynomial of multiplication-by-$\alpha(u,t)$ on the étale $\mathbb{Q}$-algebra $F(\alpha(u,t))$ of degree $2n$ factors over L as

$$\prod_{i=1}^{n} (T^2 - t_i T + u_i) \in \mathbb{Q}[T]. \tag{5.2.3}$$

The roots of this polynomial can be organized into pairs $\{\alpha_i(u,t), \beta_i(u,t)\}_{i=1,\ldots,n}$, possibly with repetition. By examining partial fraction decompositions, it follows that

$$\sum_{k \geq 0} \mathrm{Nm}(D_k(u,t)) T^k = \frac{1}{\mathrm{Nm}(\alpha(u,t) - \beta(u,t))} \sum_{\theta \in \Theta} \varepsilon(\theta) \frac{\theta}{1 - \theta T} \tag{5.2.4}$$

where

$$\Theta := \left\{ \prod_{\gamma \in S} \gamma : S \subseteq \prod_{i=1}^{n} \{\alpha_i(u,t), \beta_i(u,t)\} \right\} \tag{5.2.5}$$

and $\varepsilon(\theta)$ is 1 (resp. -1) if θ contains an even (resp. odd) number of $\beta_i(u,t)$. Therefore,

$$\sum_{m \geq 1} \mathrm{Nm}(D_{2m-2}(u,t)) T^{2m} = T^2 \frac{1}{\mathrm{Nm}(\alpha(u,t) - \beta(u,t))} \sum_{\theta \in \Theta} \varepsilon(\theta) \frac{\theta^2}{1 - \theta^2 T^2}. \tag{5.2.6}$$

The other terms of the Hilbert series (5.1.3) are plainly rational. We therefore have an algorithmic way of computing $\mathrm{Hilb}(S(\Gamma_0(\mathfrak{N})))$ as a rational function, and hence to compute $\dim S_k(\Gamma_0(\mathfrak{N}))$ for any even k. We will call the total degree of a rational function $P(T)/Q(T)$ in lowest terms the quantity $\deg P(T) + \deg Q(T)$.

Proposition 5.2.7. *The total degree of the Hilbert series is at most*

$$2 + 4(n+1) + 2^{n+1} \#\{(u,t)\}. \tag{5.2.8}$$

Proof. The Hilbert series has the explicit expression as in 5.1.3. The term with coefficient B has total degree at most $4(n+1)$ and the first term has total degree 2. Each $C(u,t)$ term is given explicitly in (5.2.6) and is a rational function with denominator of degree at most $2 \cdot \#\Theta = 2^{n+1}$ and numerator of degree 2^{n+1}. Since the expression in (5.2.6) is the same for (u,t) and $(u,-t)$, we can reduce the total degree contributed by these terms by a factor of 2. Because the total degree of $f(T) + g(T)$ is at most the sum of the total degrees of $f(T), g(T)$ for any $f(T), g(T) \in \mathbb{Q}(T)$, the result is proven. □

6. GEOMETRIC INVARIANTS

In this section, we compute geometric invariants for Hilbert modular surfaces $X(\Gamma)$ as defined in section 2.2.

6.1. **Invariants.** Let X be a smooth connected algebraic surface over $\mathbb{C}$. Regarding X as a closed oriented real 4-manifold, it has Betti numbers $b_i = \operatorname{rk} H_i(X, \mathbb{Z})$ satisfying $b_0 = b_4 = 1$, $b_1 = b_3$ and an Euler number

$$e = \sum_{i=0}^{4} (-1)^i b_i. \tag{6.1.1}$$

As a complex Kähler manifold, X admits Hodge numbers $h^{p,q} = \dim_{\mathbb{C}} H^q(X, \Omega^p)$, where Ω^p denotes the sheaf of holomorphic p-forms, which satisfy $h^{p,q} = h^{q,p}$. The geometric genus of X is $p_g = h^{0,2}$ and the irregularity is $q = h^{0,1}$. The arithmetic genus of X is $\chi = h^{0,0} - h^{0,1} + h^{0,2}$.

6.2. **Volume and Chern numbers.** We follow [17, Chapter IV]. To simplify notation, write $\Gamma(1) := \mathrm{P}\Gamma(1)_{(1)} = \mathrm{PGL}_2^+(R)$. For any discrete subgroup Γ of $\mathrm{PGL}_2^+(\mathbb{R})^2$ commensurable with $\Gamma(1)$, one can define its index as

$$[\Gamma(1) : \Gamma] = [\Gamma(1) : \Gamma(1) \cap \Gamma]/[\Gamma : \Gamma(1) \cap \Gamma] \in \mathbb{Q}. \tag{6.2.1}$$

In particular, for any fractional ideal $\mathfrak{b}$, we have $[\Gamma(1) : \mathrm{P}\Gamma(1)_{\mathfrak{b}}] = 1$. Then the volume of the quotient $\Gamma\backslash\mathcal{H}$ is given by the following formula:

$$\operatorname{vol}(\Gamma\backslash\mathcal{H}) = 2[\Gamma(1) : \Gamma]\zeta_F(-1) = -[\Gamma(1) : \Gamma]\frac{d_F^{3/2}}{\pi^{2n}}\zeta_F(2), \tag{6.2.2}$$

where ζ_F denotes the Dedekind zeta function of F. The significance of $\operatorname{vol}(\Gamma\backslash\mathcal{H})$ for the computation of invariants of $X(\Gamma)$ stems from Hirzebruch's proportionality principle [17, IV.2.1]. The next step is to compute the Chern numbers c_1^2 and c_2 of $X(\Gamma)$. In the following statement, we say that an elliptic point of $X(\Gamma)$ is of type $(n; a, b)$ if its rotation factor is (ζ_n^a, ζ_n^b) (cf. section 4.3).

Theorem 6.2.3 ([17, Theorem IV.2.5])**.** *Let $\Gamma \leq \mathrm{GL}_2^+(F)$ be a congruence subgroup. Then the Chern numbers c_1^2, c_2 of $X(\Gamma)$ are as follows:*

$$c_1^2 = 2\operatorname{vol}(\Gamma\backslash\mathcal{H}) + \sum_{\sigma \text{ cusp}} \sum_{k=1}^{r} (2 - b_{\sigma,k}) + \sum a(\Gamma; n, a, b) c(n; a, b)\,, \tag{6.2.4}$$

$$c_2 = \operatorname{vol}(\Gamma\backslash\mathcal{H}) + \ell + \sum a(\Gamma; n, a, b)\Big(\ell(n; a, b) + \frac{n-1}{n}\Big) \tag{6.2.5}$$

where

- *$b_{\sigma,1}, \dots, b_{\sigma,r}$ are the self-intersection numbers of the resolution cycle above the cusp σ,*
- *$a(\Gamma; n, a, b)$ is the number of quotient singularities of $\Gamma\backslash\mathcal{H}$ of type $(n; a, b)$;*
- *$c(n; a, b)$ is the self-intersection number of the local Chern cycle of a quotient singular of type $(n; a, b)$;*
- *$\ell(n; a, b)$ is the number of curves in the resolution of a quotient singularity of type $(n; a, b)$; and finally*
- *ℓ is the sum of the number of curves occurring in the resolution of each cusp.*

6.3. **Hodge diamond and Betti numbers.** By Noether's formula, we have $\chi = \frac{1}{12}(c_1^2 + c_2)$. Since there are no Hilbert modular forms of weight $(0,2)$ or $(2,0)$, we have $q = 0$, and so $p_g = \chi - 1$ and $h^{1,1} = e - 2\chi$.

The following strong sanity checks are available for our computations. First, the arithmetic genus χ must be integral for every congruence subgroup. Second, arithmetic genus values can be compared with dimensions of spaces of cusp forms, computed independently in section 5. Indeed, for every $\mathfrak{N}$, we must have

$$\sum_{[\mathfrak{b}] \in \mathrm{Cl}^+(R)} \chi(X_0(\mathfrak{N})_{\mathfrak{b}}) = \dim S_2(\Gamma_0(\mathfrak{N})) + h^+(R). \tag{6.3.1}$$

6.4. **Kodaira type.** If E is an exceptional curve on a surface X, i.e. E is a smooth rational curve such that $E \cdot E = -1$, then there exists a blowing-down morphism $\pi \colon X \to X'$ such that $\pi(E)$ is a point on X' and π is an isomorphism away from E. A surface is called minimal if it does not contain any exceptional curve. We recall the Kodaira classification of minimal surfaces with irregularity zero, in terms of their arithmetic genus χ and the self-intersection K^2 of their canonical divisor; see [17, VII] for details.

Kodaira dimension	type	χ	K^2
$\kappa = -1$	rational	1	$8, 9$
$\kappa = 0$	K3	2	0
	Enriques		
$\kappa = 1$	honestly elliptic	≥ 1	0
$\kappa = 2$	general type	≥ 1	≥ 1

Table 1. Enriques-Kodaira classification of minimal surfaces with $q = 0$

The computational tools developed above give a way to compute the arithmetic genus χ and K^2. Note that in order to compute K^2 of a minimal model of the surface, we need to count the number of exceptional curves on the Hilbert modular surface. In practice, we only use K^2 of the original surface as a lower bound for K^2 of the minimal model.

According to Table 1, the invariants χ and K^2 are enough to determine the Kodaira dimension, except when:

(1) $\chi = 1$, $K^2 = 8, 9$: rational or general type,
(2) $\chi = 2$, $K^2 = 0$: honestly elliptic, K3, or Enriques.

In the finitely many exceptional cases, one can compute configurations of the resolution cycles and Hirzebruch–Zagier divisors to distinguish between the various types of surfaces. We have the following criteria in cases (1) and (2) respectively.

Proposition 6.4.1 ([17, VII.2.2])**.** *If X is a smooth algebraic surface with $q = 0$ and either:*

(a) *X contains two curves C_1, C_2 such that $C_1^2 = C_2^2 = -1$ and $C_1 \cdot C_2 > 0$, or*
(b) *X contains a curve C such that $C^2 \geq 0$ and $K \cdot C < 0$,*

then X is rational.

The second criterion relies on the notion of an elliptic configuration: see [17, Def. VII.2.8].

Proposition 6.4.2 ([17, VII.2.9])**.** *Let X be a simply-connected non-rational algebraic surface. If X contains an elliptic configuration $\mathcal{C}$, then X is a blown-up K3 surface or a blown-up honestly elliptic surface. Moreover, if X also contains a (-2)-curve $D \notin \mathcal{C}$ such that $\mathcal{C} \cup \{D\}$ is connected, then X is a blown up (elliptic) K3 surface.*

We have two sources of curves on Hilbert modular surfaces: the resolution cycles, which are never exceptional, and Hirzebruch–Zagier divisors [17, Chapter V], which are exceptional if and only if they have genus 0.

6.5. **Results.** We consider Hilbert modular surfaces with $\Gamma_0^1(\mathfrak{N})_{\mathfrak{b}}$-level structure for simplicity; the other level structures could be analyzed similarly using the tools of this paper.

Table 2 shows surfaces $X_0^1(\mathfrak{N})_{\mathfrak{b}}$ with $\mathfrak{N} \neq (1)$ within our dataset in Section 7 which could be rational according to Table 1. We expect this list to exhaust rational surfaces and all of them to be rational. The intrinsic `RationalityCriterion` looks through some Hirzebruch–Zagier divisors of genus 0, blows down any subset of them, and checks the rationality criterion for these configurations of curves. In several cases, we can conclude that the surface is indeed rational; the examples with $d_F = 17$ are described in Example 7.3.4 below. In Table 2 (and Table 3 to follow), we write $\mathfrak{p}_p$ for any ideal of F above a prime $p \in \mathbb{Z}$, and the component ideal $\mathfrak{b}$ is described by its genus [17, pp. 3].

d_F	Genus of $\mathfrak{b}$	$\mathfrak{N}$
5	$+$	$\mathfrak{p}_2$, $\mathfrak{p}_5$, $\mathfrak{p}_3$, $\mathfrak{p}_{11}$, $\mathfrak{p}_2^2$, $\mathfrak{p}_{19}$, $\mathfrak{p}_2\mathfrak{p}_5$, $\mathfrak{p}_5^2$, $\mathfrak{p}_{29}$, $\mathfrak{p}_2\mathfrak{p}_{11}$, $\mathfrak{p}_{59}$
8	$+$	$\mathfrak{p}_2^2$, $\mathfrak{p}_7$, $\mathfrak{p}_2^3$, $\mathfrak{p}_2\mathfrak{p}_7$, $\mathfrak{p}_2^4$, $\mathfrak{p}_{23}$
12	$++$	$\mathfrak{p}_2$, $\mathfrak{p}_3$, $\mathfrak{p}_2^2$, $\mathfrak{p}_2\mathfrak{p}_3$, $\mathfrak{p}_3^2$, $\mathfrak{p}_{11}$
12	$--$	$\mathfrak{p}_2$, $\mathfrak{p}_3$, $\mathfrak{p}_2\mathfrak{p}_3$, $\mathfrak{p}_{11}$
13	$+$	$\mathfrak{p}_3$, $\mathfrak{p}_3^2$
17	$+$	$\mathfrak{p}_2$, $\mathfrak{p}_2^2$
24	$++$	$\mathfrak{p}_3$
28	$++$	$\mathfrak{p}_2$

TABLE 2. Hilbert surfaces $X_0^1(\mathfrak{N})_{\mathfrak{b}}$ for $\mathfrak{N} \neq (1)$ satisfying $\chi = 1$

In future work, we hope to check that the other surfaces in the table are indeed rational, by including more Hirzebruch–Zagier divisors, the divisors coming from resolutions of elliptic points, and computing $K \cdot F_N$ by using [17, Corollary 4.1] in order to get F_N^2 in cases when F_N has higher genus.

Similarly, Table 3 shows surfaces $X_0^1(\mathfrak{N})_{\mathfrak{b}}$ with $\mathfrak{N} \neq (1)$ within our dataset in Section 7 which could be honestly elliptic, K3, or Enriques according to Table 1. We also expect this list to be exhaustive. Similarly to our implementation of the rationality criterion, one could implement the criterion for X to be a blown-up K3 surface. This would require describing all the Hirzebruch–Zagier divisors on higher level Hilbert modular surfaces and using [17, Corollary 4.1] to compute their self-intersection numbers. We hope to finish the classification of all Hilbert modular surfaces with Γ_0 and Γ_1 level structures in future work.

d_F	Genus of $\mathfrak{b}$	$\mathfrak{N}$
5	$+$	$\mathfrak{p}_{31}$
8	$+$	$\mathfrak{p}_3$, $\mathfrak{p}_{17}$, $\mathfrak{p}_2^5$
12	$++$	$\mathfrak{p}_2^3$, $\mathfrak{p}_2^2\mathfrak{p}_3$, $\mathfrak{p}_2^4$
12	$--$	$\mathfrak{p}_2^2$, $\mathfrak{p}_2^3$, $\mathfrak{p}_2^2\mathfrak{p}_3$
13	$+$	$\mathfrak{p}_2$, $\mathfrak{p}_3\overline{\mathfrak{p}_3} = (3)$
17	$+$	$\mathfrak{p}_2\overline{\mathfrak{p}_2} = (2)$, $\mathfrak{p}_2^3$
21	$++$	$\mathfrak{p}_3$, $\mathfrak{p}_2$, $\mathfrak{p}_5$, $\mathfrak{p}_7$, $\mathfrak{p}_3^2$
21	$--$	$\mathfrak{p}_3$, $\mathfrak{p}_5$
24	$++$	$\mathfrak{p}_2$, $\mathfrak{p}_2^2$
24	$--$	$\mathfrak{p}_2$
28	$++$	$\mathfrak{p}_3$, $\mathfrak{p}_2^2$, $\mathfrak{p}_3^2$
28	$--$	$\mathfrak{p}_3$
33	$++$	$\mathfrak{p}_2$, $\mathfrak{p}_3$, $\mathfrak{p}_2^2$
33	$--$	$\mathfrak{p}_2$

Table 3. Hilbert surfaces $X_0^1(\mathfrak{N})_{\mathfrak{b}}$ for $\mathfrak{N} \neq (1)$ satisfying $\chi = 2$ and $K^2 \leq 0$

7. DATA COMPUTED

The data resulting from our computations is publicly available online at `https://github.com/edgarcosta/hilbertmodularsurfacesdata/`. We hope to include it in the LMFDB in the near future.

7.1. **Scope of data.** We computed geometric invariants χ and K^2 (hence the Hodge diamond) for all Hilbert surfaces attached to congruence subgroups $\Gamma_0(\mathfrak{N})_{\mathfrak{b}}$ and $\Gamma_0^1(\mathfrak{N})_{\mathfrak{b}}$ for real quadratic fields F with discriminant $d_F \leq 3000$ and levels $\mathfrak{N}$ with $\mathrm{Nm}(\mathfrak{N}) \leq 5000/{d_F}^{3/2}$, where $\mathfrak{b}$ ranges over all possible narrow classes in $\mathrm{Cl}^+(R)$. (This type of cutoff is meant as a rough approximation for the volume of the surface.) We find 4517 possibilities for $(F, \mathfrak{N}, \mathfrak{b})$ in this range. Table 4 displays timings for computing the geometric invariants and cusp resolutions for Γ_0 and Γ_0^1.

	Γ_0	Γ_0^1
invariants	104.46	92.36
cusps	53.99	52.53

Table 4. Timings for the various computations, given in CPU minutes

We also computed the Hilbert series for the dimensions of cuspidal spaces $S_k(\Gamma_0(\mathfrak{N}))$ in the above range, but omitted some examples that took less than 3 hours due to the skewness of the Minkowski lattice of F. We hope to optimize these skewed cases in the future.

7.2. **Features of the data.** Table 5 presents the distribution of the different surface types in our dataset. Of the 4517 surfaces $X_0(\mathfrak{N})_{\mathfrak{b}}$ (resp., $X_0^1(\mathfrak{N})_{\mathfrak{b}}$), in all but 184 (resp., 175) cases the invariants we computed allowed to completely determine the surface's Kodaira dimension. We leave a more detailed analysis of the data for future work.

Type	Kodaira dimension	Γ_0	Γ_0^1
rational	$\kappa = -1$	18	15
honestly elliptic	$\kappa = 1$	7	16
general type	$\kappa = 2$	4308	4311
unknown	$\kappa \in \{-1, 2\}$	61	44
	$\kappa \in \{0, 1\}$	5	12
	$\kappa \in \{0, 1, 2\}$	51	43
	$\kappa \in \{1, 2\}$	67	76
	total	184	175

Table 5. Counts for the number of Hilbert modular surfaces in our dataset of each Kodaira dimension

7.3. **Examples.** Below we present some interesting examples of surfaces that we observed in our dataset.

Example 7.3.1. Let $F = \mathbb{Q}(\sqrt{85})$ and $\mathfrak{N} = (1)$. Then $h = h^+ = 2$; let $\mathfrak{b} = (3, 1 + \sqrt{85})$ be a representative for the nontrivial narrow class.

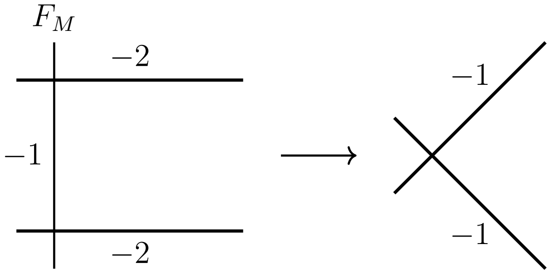
For $X_0(\mathfrak{N})_{(1)}$, we calculate that $\chi = 4$. We initially compute $K^2 = -8$ using (6.2.4); however our model for the surface is not minimal. We find 8 exceptional Hirzebruch-Zagier divisors and, blowing these down, instead find that $K^2 = 0$. From the results in Table 1, this allows us to conclude that $\kappa = 1$ and hence the surface is honestly elliptic.

For the other component $X_0(\mathfrak{N})_{\mathfrak{b}}$, we compute that $\chi = 4$ and initially find that $K^2 = 0$. Again our model is not minimal and this time we find 2 exceptional Hirzebruch-Zagier divisors, which lead to a corrected computation of $K^2 = 2$. This implies that $\kappa = 2$ and the surface is of general type. Thus we have found a Hilbert modular surface whose components belong to different Kodaira classes, and hence are not isomorphic or even birational.

Example 7.3.2. Let $F = \mathbb{Q}(\sqrt{11})$ and $\mathfrak{N} = (1)$. Then $h = h^+ = 2$; let $\mathfrak{b} = (3 - \sqrt{11})$ be a representative for the nontrivial narrow class. For $X_0^1(\mathfrak{N})_{(1)}$, we calculate that $\chi = 2$. The presence of 8 exceptional Hirzebruch-Zagier divisors leads to $K^2 = -8 + 8 = 0$. This time we are unable to determine the Kodaira dimension exactly, as consulting Table 1 only tells us that $\kappa \in \{0, 1\}$. For $X_0^1(\mathfrak{N})_{\mathfrak{b}}$, we have $\chi = 3$ and $K^2 = -2 + 2 = 0$. Thus $\kappa = 1$ and the surface is honestly elliptic.

Example 7.3.3. Let $F = \mathbb{Q}(\sqrt{165})$ and $\mathfrak{N} = (1)$. Then $\mathrm{Cl}(R) \cong C_2$ and $\mathrm{Cl}^+(R) \cong C_2 \times C_2$, where C_2 is the cyclic group of order 2. Computing for $X_0^1(\mathfrak{N})_{(1)}$, we find that $\chi = 4$. The existence of 20 exceptional Hirzebruch-Zagier divisors gives $K^2 = -20 + 20 = 0$. Thus $\kappa = 1$ and the surface is honestly elliptic. By contrast, for $X_0(\mathfrak{N})_{(1)}$ we have $\chi = 3$ and $K^2 = -10 + 20 = 10$, and hence $\kappa = 2$ and the surface is of general type. (We note in passing that, for both X_0 and X_0^1, the remaining 3 components are all of general type.)

Example 7.3.4. When $F = \mathbb{Q}(\sqrt{17})$, we can show that all the Hilbert surfaces with $\chi = 1$ are rational. By Table 2, we only need to consider the cases $\mathfrak{N} = \mathfrak{p}_2^i$, $i = 1, 2$. In each case, we let $M = 2^i$ and consider the exceptional Hirzebruch–Zagier divisor F_M. We compute that we have the following configuration of curves in both cases, where the thick lines are curves in the resolutions of the cusps.



We blow down the exceptional curve F_M and apply Proposition 6.4.1 to conclude that the surface is rational.

7.4. **Future directions.** An immediate goal is to gather data for other standard congruence subgroups, namely $\Gamma_1(\mathfrak{N})_{\mathfrak{b}}$, $\Gamma_1^1(\mathfrak{N})_{\mathfrak{b}}$, and $\Gamma(\mathfrak{N})_{\mathfrak{b}}$. The case of $\Gamma_1(\mathfrak{N})_{\mathfrak{b}}$ for non-squarefree levels seems to have additional complications, in that the stabilizer of a cusp does not seem to be of the exact form $G(M,V)$, and only contains such a group with finite index. It remains to determine how this finite quotient acts on the resolution cycles.

We also hope complete the classification in Kodaira types of the surfaces we have computed and resolve any ambiguities. We aim to determine each surface's Kodaira dimension exactly by extending our results on Hilbert series from section 5. This involves studying algorithmically which cusp forms of higher weights extend to the resolution cycles above elliptic points. We also plan to further study Hirzebruch-Zagier divisors on higher level modular surfaces in order to extend our rationality criterion and to distinguish between honestly elliptic, K3, and Enriques surfaces.

Van der Geer proves [17, Theorem VII.3.3] that the Hilbert modular surface $X_0^1(1)_{\mathfrak{b}}$ over a real quadratic field F is of general type in all but finitely many cases. A key result in proving this theorem are estimates [17, Theorem VII.5.1] which show that if $d_F > 500$, then $X_0^1(1)_{\mathfrak{b}}$ is of general type. It seems that an analogous result should be true in our more general setting of higher level and further variants ($\Gamma_0, \Gamma_0^1, \Gamma_1, \Gamma_1^1$, etc.) of Hilbert modular groups. Proving such a result would guarantee that Table 2 and Table 3 are exhaustive.

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Department of Mathematics, Dartmouth College, 6188 Kemeny Hall, Hanover, NH 03755, USA
Email address: `assaferan@gmail.com`
URL: http://www.math.dartmouth.edu/~eassaf/

Department of Mathematics & Statistics, McMaster University, Hamilton Hall, 1280 Main Street West, Hamilton, ON, L8S 4K1, Canada
Email address: `babeia@mcmaster.ca`
URL: https://angelicababei.com

School of Mathematics and Statistics, Clemson University, Clemson, SC, 29631
Email address: `benjaminkbreen@gmail.com`
URL: www.benbreenmath.com

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, USA
Email address: `edgarc@mit.edu`
URL: https://edgarcosta.org

Department of Mathematics, Dartmouth College, 6188 Kemeny Hall, Hanover, NH 03755, USA
Email address: `juanita.gr@dartmouth.edu`
URL: https://math.dartmouth.edu/~jduque/

Mathematical Institute, University of Oxford, Woodstock Road, Oxford, OX2 6GG, UK
Email address: `horawa@maths.ox.ac.uk`
URL: https://people.maths.ox.ac.uk/horawa/

Department of Mathematics, Harvard University, 1 Oxford St., Cambridge, MA 02138, USA
Email address: `kieffer@math.harvard.edu`
URL: https://scholar.harvard.edu/kieffer

Department of Mathematics, Dartmouth College, 6188 Kemeny Hall, Hanover, NH 03755, USA
Email address: `avinash.a.kulkarni@dartmouth.edu`
URL: https://math.dartmouth.edu/~akulkarn/

Department of Mathematics, Dartmouth College, 6188 Kemeny Hall, Hanover, NH 03755, USA
Email address: `Grant.S.Molnar.GR@dartmouth.edu`
URL: https://www.grantmolnar.com

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, USA
Email address: `sschiavo@mit.edu`
URL: https://math.mit.edu/~sschiavo/

Department of Mathematics, Dartmouth College, 6188 Kemeny Hall, Hanover, NH 03755, USA
Email address: `jvoight@gmail.com`
URL: http://www.math.dartmouth.edu/~jvoight/