A DATABASE OF PARAMODULAR FORMS FROM QUINARY ORTHOGONAL MODULAR FORMS

ERAN ASSAF, WATSON LADD, GUSTAVO RAMA, GONZALO TORNARÍA, AND JOHN VOIGHT

ABSTRACT. We compute tables of paramodular forms of degree two and cohomological weight via a correspondence with orthogonal modular forms on quinary lattices.

1. INTRODUCTION

Number theorists have a longstanding tradition of making tables of modular forms (for a brief history, see [CMFs21, §2]), with myriad applications to arithmetic and geometry. Moving beyond GL_2, there has been substantial interest in similar catalogues of Siegel modular forms, as computed first for $\mathrm{Sp}_4(\mathbb{Z})$ by Kurokawa [Kur78] and then more systematically by Skoruppa [Sko92], Raum [Rau10], and others. Moving beyond trivial level, we find several interesting families of congruence subgroups of symplectic groups. Among these possibilities, the paramodular groups have recently received considerable interest, owing in part to their agreeable theory of newforms [RS07] as well as applications in the Langlands program [BK19, Gro16]. Direct computations of paramodular forms have focused on the more troublesome case of (noncohomological) weight 2 [PY15, BPY16, PSY17] (analogous to weight 1 classical modular forms), working with Fourier expansions using clever and sophisticated techniques.

In this paper, we report on our computation of a moderately large database of paramodular forms for GSp_4 (i.e., degree 2), but via a complementary approach in weight ≥ 3: we compute with algebraic modular forms [Gro99] on orthogonal groups of positive definite quadratic forms in five variables. Instead of working with Fourier expansions, we access only the underlying systems of Hecke eigenvalues (enough for Galois representations and L-function). An explicit correspondence between orthogonal and paramodular forms was first conjectured by Ibukiyama [IK17, Ibu19] and recently proven by van Hoften [vH21], Rösner–Weissauer [RW21], and Dummigan–Pacetti–Rama–Tornaría [DPRT21]. Our approach using quinary quadratic forms is analogous to the use of ternary quadratic forms to compute classical modular forms introduced by Birch [Bir91] (see also [Tor05, Ram14, Hei16, HTV]). Our algorithms involve lattice methods, as described by Greenberg–Voight [GV14] and further developed and investigated by Hein [Hei16], Ladd [Lad18], and Rama [Ram20, Ram20git].

The tables computed here supersede data computed by Rama–Tornaría [RT20] in squarefree level $N \leq 1000$: we compute systematically in nonsquare level $N \leq 1000$, in higher weight, and with more Dirichlet coefficients. Our data is available at `https://github.com/assaferan/omf5_data`, and we will incorporate it into the L-functions and Modular Forms Database (LMFDB) [LMFDB].

Date: January 22, 2023.

Acknowledgements. The authors would like to thank Ariel Pacetti for helpful conversations. Assaf and Voight were supported by a Simons Collaboration grant (550029, to JV). Rama and Tornaría were partially supported by CSIC I+D 2020/651.

2. Algorithmic comments

Definite orthogonal methods for paramodular forms. The relation between modular forms of SO(5) and automorphic forms on GSp(4) with trivial central character is predicted by Langlands functoriality. An explicit correspondence was conjectured by Ibukiyama in [IK17, Ibu19] involving two steps: a correspondence between (para)modular forms of GSp(4) and its compact twist $\mathrm{GU}(2, B)$, where B is a definite quaternion algebra; and a correspondence between modular forms of $\mathrm{GU}(2, B)$ and those of $\mathrm{SO}(Q)$ for a suitable chosen quinary quadratic form Q. The first correspondence was proved by van Hoften [vH21] and Rösner–Weissauer [RW21], and extended by Dummigan–Pacetti–Rama–Tornaría [DPRT21] where the second correspondence was proved.

More precisely [DPRT21, Theorem 9.8], we can compute the space $S^{\mathrm{new}}_{k,j}(K(N))$ of paramodular newforms of level N and weight (k, j) under the following assumptions:

1. There is a prime p_0 such that $p_0 \parallel N$ (p exactly divides N); and
2. $k \geq 3$ and $j \geq 0$ even.

Suppose that these conditions hold. Then [DPRT21, Theorem 5.14] there exists a unique genus of positive definite integral quinary quadratic forms of (half-)determinant N with Eichler invariants -1 at p_0 and $+1$ at all other primes [DPRT21, §5]. We choose one quadratic form $Q = Q_{N,p_0}$ in the genus, corresponding to a lattice Λ, noting that the corresponding space of orthogonal modular forms is independent of this choice.

Following notation in [DPRT21, §8], write $a := k + j - 3$ and $b := k - 3$, noting $a \equiv b \pmod 2$. Let $W_{a,b}$ be the corresponding weight representation of $\mathrm{SO}_5(\mathbb{C})$; and for each $d \parallel N$, let θ_d be the associated spinor character. Let $M_{a,b}(\mathrm{SO}(\Lambda), \theta_d)$ for the space of orthogonal modular forms for Q_N with weight representation $W_{a,b}$ and character θ_d.

Theorem 2.1 ([DPRT21, Theorem 9.8]). *There is an isomorphism of Hecke modules*

$$S^{p_0\text{-}new}_{k,j}(K(N))_{\mathbf{(G)}} \simeq \bigoplus_{d \parallel N} M_{a,b}(\mathrm{SO}(\Lambda), \theta_d)_{\mathbf{(G)}}$$

between the space of p_0-new paramodular cusp forms of weight (k, j) of general type **(G)** *and the space of orthogonal modular forms for Λ with arbitrary spinor character of general type* **(G)**.

Moreover, the paramodular forms corresponding to $M_{a,b}(\mathrm{SO}(\Lambda), \theta_d)_{\mathbf{(G)}}$ have their Atkin-Lehner signs ε_p determined by d: we have $\varepsilon_{p_0} = -1$ if and only if $p_0 \nmid d$, and for primes $p \neq p_0$ we have $\varepsilon_p = -1$ if and only if $p \mid d$.

The condition **(G)** of general type concerns the type of the automorphic representation, following Schmidt [Sch18, §1.1]: it excludes the forms of Saito–Kurokawa type **(P)** on both sides and the forms of Yoshida type **(Y)** appearing as orthogonal modular forms. See the section on detecting lifts below for further discussion.

We can compute the Hecke operators for all good primes $p \nmid N$ and bad primes $p \parallel N$ [RT20]. In either case, the Euler factors can be obtained by computing p and p^2-neighbors. See the section on bad Euler factors below for bad primes $p^2 \mid N$.

Algorithm overview. Algorithms to compute with orthogonal modular forms using lattice methods were exhibited by Greenberg–Voight [GV14]; a recent overview is given by Assaf–Fretwell–Ingalls–Logan–Secord–Voight [AFILSV22, §3]. These algorithms take as input a positive definite quadratic form and compute the action of Hecke operators on spaces of functions on the class set, with values in a weight representation. The Hecke operators are computed as p-neighbors, after Kneser.

Algorithmic improvements. In order to make the calculations of Hecke operators mentioned in the previous section more efficient, it is possible to take advantage of the action of the isometry group of the lattice. Indeed, two p-isotropic vectors in the same orbit of the isometry group $\mathrm{Aut}(\Lambda)$ will produce the same target lattice when applying the p-neighbor relation. It is also possible to save on isometry testing via taking the first few entries of the theta series for the lattice.

To carry out this idea (which has been observed before), We present a few algorithmic improvements to further speed up the computation. First, instead of precomputing all the orbits of isotropic vectors under $\mathrm{Aut}(\Lambda)$, we order the isotropic vectors in a lexicographical order. Given a $\mathbb{Z}_p$-isotropic vector v, we compute its orbit under $\mathrm{Aut}(\Lambda)$, and proceed with the computation only when it is the minimal vector in its orbit.

Second, we precompute the automorphism group of all lattices in the genus, and their conjugations into a single quadratic space, saving the cost of conjugation when computing the spinor norm.

Finally, for a given genus we measure the cost T_{isom} of isometry testing and the cost $T_\theta(B)$ of computing the short vectors of length up to B, and we choose B that optimizes the total cost $\alpha(B)T_{\mathrm{isom}} + T_\theta(B)$, where $\alpha(B)$ is the average number of collisions in the hash table, with the lattices in the genus averaged by the size of their automorphism groups. This follows from the fact that the frequency of appearance of a lattice Λ as a p-neighbor is inversely proportional to $\#\,\mathrm{Aut}(\Lambda)$, as proven by Chenevier [Che22].

Detecting lifts. Among the orthogonal forms we compute, like Eisenstein series in the classical case, we have forms that arise from lifts (endoscopy). Since those lifts are classified and can be computed in other ways, we focus on what remains on computing newforms of type **(G)**.

To discard forms of type **(P)** corresponding to Saito–Kurokawa lifts, a single good Euler factor is enough: a form of type **(P)** will have a Satake parameter $p^{1/2}$ that cannot otherwise appear by the Ramanujan conjecture for non-CAP forms [Wei09].

To discard forms of type **(Y)** corresponding to Yoshida lifts, it suffices to find that a single good Euler factor is irreducible. If all the computed degree 4 Euler factors are product of two degree 2 factors, we look in tables to conjecturally identify the form as a Yoshida lift, as we know exactly which Yoshida lifts should appear [DPRT21, Proposition 9.1]. To identify the lifts, a simple inspection and comparison of the traces often suffices.

Newforms and oldforms. In all cases, we have a good guess as to what is new and old; computing inductively we can verify that a form is not a lift and not an oldform. However, old lifts and nonlifts may appear in the space of orthogonal modular forms with multiplicity, so we need to know if there are newforms that look like lifts or oldforms up to the precision computed, in which case we can increase the precision (compute more Hecke eigenvalues). In this way, we rigorously compute a subset of newforms, but need additional certification to be sure that the forms that look like oldforms are in fact old (and do not just agree with oldforms to the precision computed).

To improve upon this, we refer to the local newform theory of Roberts–Schmidt [RS07], which gives precise formulas for the multiplicity of paramodular oldforms for forms of types **(G)** and **(Y)** [RS07, Theorem 7.5.6] as well as type **(P)** [RS07, Theorem 5.5.9]. We plan to analyze this multiplicity and implement degeneracy maps in future work, to certify our list of oldforms.

3. Running the calculation

Data and running time. For reliability, we carried out and compared two separate implementations to compute the data, one in C and one in PARI/GP. Eventually, these gave the same output. The code can be found at `https://github.com/assaferan/modfrmalg` and `https://gitlab.fing.edu.uy/grama/quinary`.

We computed the spaces of paramodular forms of level N and weight (k,j), the Hecke eigenforms and the eigenvalues of the Hecke operators in the following ranges:

- $(k,j)=(3,0)$, $D=N\leq 1000$, good $T_{p,i}$ with $p^i<200$
- $(k,j)=(4,0)$, $D=N\leq 1000$, good $T_{p,i}$ with $p^1<100$, $p^2<30$
- $(k,j)=(3,2)$, $D=N\leq 500$, good $T_{p,i}$ with $p^1<100$, $p^2<30$

The total counts are summarized as follows:

(k,j)	Newspaces			Newforms		
	squarefree N	nonsquarefree N	total	squarefree N	nonsquarefree N	total
$(3,0)$	2 764	4 817	7 581	52 181	23 853	76 034
$(3,2)$	1 363	3 072	4 435	72 551	29 226	101 777
$(4,0)$	2 856	7 783	10 639	287 974	132 380	420 354

Table 1. Newspace and newform data computed

For Hecke irreducible spaces of dimension ≥ 20, we only store the traces; to avoid painful calculations in an extension field, we compute modulo a prime $\mathfrak{p}$ of degree 1 which is large enough, and reconstruct.

The total data takes approximately 200 MB of disk space and took a total of 4575 hours of CPU time on a standard processor.

Bad Euler factors. When $p^2\mid N$, the local Euler factor has degree at most 2 and is given by Roberts–Schmidt [RS07, Theorem 7.5.3] in terms of eigenvalues for a pair of Hecke operators which correspond to p and p^2-neighbors as in [DPRT21, Proposition 8.5] (extended in the same way for $p\neq p_0$). In practice, for $p^2\mid N$ we found easier to guess the appropriate Hecke operator via reconstruction by checking that the functional equation for the L-function is satisfied (ruling out all possibilities but one).

The first such bad factor occurs at level $76=2^2\cdot 19$: the bad Euler factors are

$$L_2(f_{76},X)=1+5X+2^3X^2$$
$$L_{19}(f_{76},X)=(1+19X)(1-50X+19^3X^2)$$

The next one occurs at level $96 = 2^5 \cdot 3$, and its Euler factors at the bad primes are

$$L_2(f_{96}, X) = 1 + 4X + 2^3X^2$$
$$L_3(f_{96}, X) = (1 - 3X)(1 + 8X + 3^3X^2).$$

The functional equations have been verified to a precision of 22 decimal digits.

References

[AFILSV22] Eran Assaf, Dan Fretwell, Colin Ingalls, Adam Logan, Spencer Secord, and John Voight, *Definite orthogonal modular forms: computations, excursions, and discoveries*, Res. Number Theory **8** (2022), no.-4, Paper no. 70, 32.

[Bir91] B. J. Birch, *Hecke actions on classes of ternary quadratic forms*, Computational number theory (Debrecen, 1989), de Gruyter, Berlin, 1991, 191–212.

[BPY16] Jeffery Breeding, II, Cris Poor, and David S. Yuen, *Computations of spaces of paramodular forms of general level*, J. Korean Math. Soc. **53** (2016), no. 3, 645–689.

[Che22] Gaëtan Chenevier, *Statistics for Kneser p-neighbors* (with an appendix by O. Taïbi), Bull. Soc. Math. France **150** no.-3, 473-516 (2022).

[CMFs21] Alex J. Best, Jonathan Bober, Andrew R. Booker, Edgar Costa, John Cremona, Maarten Derickx, David Lowry-Duda, Min Lee, David Roe, and Andrew V. Sutherland, *Computing classical modular forms*, Arithmetic geometry, number theory, and computation, eds. Jennifer S. Balakrishnan, Noam Elkies, Brendan Hassett, Bjorn Poonen, Andrew V. Sutherland, and John Voight, Simons Symp., Springer, Cham, 2021, 131–213.

[BK19] Armand Brumer and Kenneth Kramer, *Paramodular abelian varieties of odd conductor*, Trans. Amer. Math. Soc. **366** (2014), no. 5, 2463–2516; and *Corrigendum to "Paramodular abelian varieties of odd conductor"*, Trans. Amer. Math. Soc. **372** (2019), no. 3, 2251–2254.

[DPRT21] Neil Dummigan, Ariel Pacetti, Gustavo Rama, and Gonzalo Tornaría, *Quinary forms and paramodular forms*, preprint, 2021, `arXiv:2112.03797`.

[GV14] Matthew Greenberg and John Voight, *Lattice methods for algebraic modular forms on classical groups*, Computations with modular forms, eds. Gebhard Boeckle and Gabor Wiese, Contrib. Math. Comput. Sci., vol. 6, Springer, Berlin, 2014, 147–179.

[Gro99] Benedict H. Gross, *Algebraic modular forms*, Israel J. Math. **113** (1999), 61–93.

[Gro16] B. Kh. Gross, *On the Langlands correspondence for symplectic motives*, Izv. Ross. Akad. Nauk Ser. Mat. **80** (2016), no. 4, 49–64; translation in Izv. Math. **80** (2016), no. 4, 678–692.

[Hei16] Jeffrey Hein, *Orthogonal modular forms: An application to a conjecture of birch, algorithms and computations*, PhD Thesis, Dartmouth College, 2016, https://doi.org/10.1349/ddlp.2156.

[HTV] Jeffery Hein, Gonzalo Tornaría, and John Voight, *Hilbert modular forms as orthogonal modular forms*, 2023, preprint.

[IK17] Tomoyoshi Ibukiyama and Hidetaka Kitayama, *Dimension formulas of paramodular forms of squarefree level and comparison with inner twist*, J. Math. Soc. Japan **69** (2017), no. 2, 597–671.

[Ibu19] Tomoyoshi Ibukiyama, *Quinary lattices and binary quaternion hermitian lattices*, Tohoku Math. J. (2) **71** (2019), no. 2, 207–220.

[Kur78] Nobushige Kurokawa, *Examples of eigenvalues of Hecke operators on Siegel cusp forms of degree two*, Invent. Math. **49** (1978), no. 2, 149–165.

[Lad18] Watson Bernard Ladd, *Algebraic modular forms on* $\mathrm{SO}_5(\mathbb{Q})$ *and the computation of paramodular forms*, PhD thesis, University of California, Berkeley, 2018.

[LMFDB] The LMFDB Collaboration, *The L-functions and modular forms database*, http://www.lmfdb.org, 2022.

[PSY17] Cris Poor, Jerry Shurman, and David S. Yuen, *Siegel paramodular forms of weight 2 and squarefree level*, Int. J. Number Theory **13** (2017), no. 10, 2627–2652.

[PY15] Cris Poor and David S. Yuen, *Paramodular cusp forms*, Math. Comp. **84** (2015), 1401–1438.

[RT20] Gustavo Rama and Gonzalo Tornaría, *Computation of paramodular forms*, ANTS XIV: Proceedings of the Fourteenth Algorithmic Number Theory Symposium, ed. Steven D. Galbraith, Open Book Ser., vol. 4, Math. Sci. Publ., Berkeley, CA, 2020, 353–370.

[Ram14] Gustavo Rama, *Módulo de Brandt generalizado*, MSc Thesis, Universidad de la República, 2014, http://cmat.edu.uy/cnt/msc.2014.0001.

[Ram20] Gustavo Rama, *Cálculo de formas paramodulares utilizando formas modulares ortogonales de* $O(5)$, PhD Thesis, Universidad de la República, 2020, http://cmat.edu.uy/cnt/phd.2021.0001.

[Ram20git] Gustavo Rama, Quinary orthogonal modular forms code repository, 2020, https://gitlab.fing.edu.uy/grama/quinary.

[Rau10] Martin Raum, *Efficiently generated spaces of classical Siegel modular forms and the Böcherer conjecture*, J. Aust. Math. Soc. **89** (2010), no. 3, 393–405.

[RS07] Brooks Roberts and Ralf Schmidt, *Local newforms for* $\mathrm{GSp}(4)$, Lecture Notes in Math., vol. 1918, Springer, Berlin, 2007.

[RW21] Mirko Rösner and Rainer Weissauer, *Global liftings between inner forms of* $\mathrm{GSp}(4)$, preprint, 2021, arXiv:2103.14715.

[Sch18] Ralf Schmidt, *Packet structure and paramodular forms*, Trans. Amer. Math. Soc. **370** (2018), no. 5, 3085–3112.

[Sko92] Nils-Peter Skoruppa, *Computations of Siegel modular forms of genus two*, Math. Comp. **58** (1992), no. 197, 381–398.

[Tor05] Gonzalo Tornaría, *The Brandt module of ternary quadratic lattices*, PhD Thesis, University of Texas at Austin, 2005, http://hdl.handle.net/2152/2129.

[vH21] Pol van Hoften, *A geometric Jacquet-Langlands correspondence for paramodular Siegel threefolds*, Math. Z. **299** (2021), no. 3–4, 2029–2061.

[Wei09] Rainer Weissauer, *Endoscopy for* $\mathrm{GSp}(4)$ *and the cohomology of Siegel modular threefolds*, Lecture Notes in Math., vol. 1968, Springer-Verlag, Berlin, 2009.

Department of Mathematics, Dartmouth College, 6188 Kemeny Hall, Hanover, NH 03755, USA
Email address: assaferan@gmail.com
URL: http://www.math.dartmouth.edu/~eassaf/

Akamai Technologies
Email address: watsonbladd@gmail.com

Facultad de Ingeniería, Universidad de la República, Montevideo, Uruguay
Email address: grama@fing.edu.uy

Centro de Matemática, Universidad de la República, Montevideo, Uruguay
Email address: tornaria@cmat.edu.uy

Department of Mathematics, Dartmouth College, 6188 Kemeny Hall, Hanover, NH 03755, USA
Email address: jvoight@gmail.com
URL: http://www.math.dartmouth.edu/~jvoight/