

ERRATA:
CHARACTERIZING QUATERNION RINGS
OVER AN ARBITRARY BASE

JOHN VOIGHT

This note gives errata and addenda for the article *Characterizing quaternion rings over an arbitrary base* [2].

ERRATA

- (1) We use the term *isometry* when the term *similarity* is probably better for our definition of morphisms of quadratic forms. A similarity between two free quadratic modules is required to have invertible determinant—rather than an isometry, which is presumed to have determinant 1). It would have been clearer to use the word *similarity* in our context.
- (2) Balmer and Calmès [1] have also studied a notion similar to our notion of a parity factorization (over schemes over $\mathbb{Z}[1/2]$): they call it a *quadratic alignment*.
- (3) In Lemma 1.5, the hypothesis $\text{rk}(B) > 2$ (not yet imposed) was omitted.
- (4) Before Lemma 2.1, B/R is claimed to be a (faithfully) projective R -module. For this, a lemma is used which assumes that R is noetherian, but this is not used in the proof; the result is correct as stated by noetherian reduction.
- (5) §3 “An equivalence of categories”, and end of the proof of Theorem 3.13: it is claimed that an equivalence of categories has been proven; this has not been shown, and it is not stated in the theorem. (We have shown that the even Clifford functor is essentially surjective and full, but it need not be faithful; to obtain an equivalence, one would at least need to include anti-isomorphisms.)
- (6) Example 3.4: should be $q_{\max}(x, y, z) = x^2 + y^2 + z^2 + xy + xz$.
- (7) Page 128, paragraph 2: The reference “Knus [11, §III.3]” should be “Knus [13, §III.3]”.
- (8) Page 132, last paragraph: In the definition of crossed product, it is hazardous to use b for the bilinear form since it also is used as a coefficient in the multiplication table (Q). Also, the definition of crossed product is incorrect. Instead of assuming that J is an invertible S -module, one should instead only assume that J is projective of rank 2 as an R -module and is a *traceable* S -module [3, p. 1758], so that the trace maps $\text{Tr} : S \rightarrow R$ induced by multiplication on S and on J are equal. (Further comments and remarks are made in the revised version.)

ADDENDA

- (1) Bjorn Poonen pointed out that the parity factorization can be more simply described. We are solving the equation $P^{\otimes 2} \otimes Q \simeq R$ with R fixed. In this equation, it is enough to remember P , since there is then a unique solution Q .

The notion of parity factorization can also be replaced by remembering more simply a class in $\text{Pic}(R)$ (suggested by Bjorn Poonen). That is to say, there is a bijection

$$\left\{ \begin{array}{l} \text{Isomorphism classes of quaternion} \\ \text{rings } B \text{ over } R \text{ equipped with a parity} \\ \text{factorization } p : P^{\otimes 2} \otimes Q \xrightarrow{\sim} \wedge^4 B \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{quaternion rings } B \text{ over } R \\ \text{equipped with } [P] \in \text{Pic}(R) \end{array} \right\}$$

which is functorial with respect to the base ring R : to a quaternion ring B equipped with a parity factorization $p : P^{\otimes 2} \otimes Q \xrightarrow{\sim} \wedge^4 B$, we associate the class $[P] \in \text{Pic}(R)$, and conversely given B with $[P]$ we take the parity factorization

$$p : P^{\otimes 2} \otimes (\wedge^4 B \otimes (P^\vee)^{\otimes 2}) \xrightarrow{\sim} \wedge^4 B.$$

REFERENCES

- [1] Paul Balmer and Baptiste Calmès, *Bases of total Witt groups and lax-similitude*, J. Algebra Appl. **11** (2012), no. 3, 1250045, 24 pp.
- [2] John Voight, *Characterizing quaternion rings over an arbitrary base*, J. Reine Angew. Math. **657** (2011), 113–134.
- [3] Melanie Wood, *Gauss composition over an arbitrary base*, Adv. Math. **226** (2011), no. 2, 1756–1771.