ERRATA: *ON THE ARITHMETIC DIMENSION OF TRIANGLE GROUPS*

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This note gives errata for the article *On the arithmetic dimension of triangle groups* [1]. The authors thank Cherng-tiao Perng.

(1) Lemma 1.9: the cardinality signs are missing, so it should read
$$\begin{aligned}\mathrm{adim}(a,b,c) &= \#\{k \in (\mathbb{Z}/2m\mathbb{Z})^\times/H : \sigma_k(\beta) > 0\} \\ &= \#\{k \in (\mathbb{Z}/2m\mathbb{Z})^\times/H : \kappa(a,b,c;k) < 0\}.\end{aligned}$$

(2) Lemma 3.4: there is a typo in the inequality, and the statement should read: $\kappa(a,b,c;k) \geq 0$ if and only if (3.5) holds. In the proof, we let $f(z) = \kappa(a,b,c;k)$ and considered when $f(z) \geq 0$. The remaining statements using Lemma 3.4 (Corollary 3.6, Proposition 4.6, Theorem 5.2) use the corrected Lemma 3.4; before (3.8) the inequality should be switched to > 0; the rest of the paper remains unchanged.

(3) Proof of Lemma 3.4, "The discriminant simplifies as": This should read
$$\sqrt{t^2-4n} = \sqrt{4\cos^2\frac{k_a\pi}{a}\cos^2\frac{k_b\pi}{b} - 4\cos^2\frac{k_a\pi}{a} - 4\cos^2\frac{k_b\pi}{b} + 4}$$
(so the final -4 should be $+4$), as shown on the next line.

(4) Proof of Proposition 4.6, "In particular $1 \leq 3q/a$": Should be "In particular, $1 < 3q/a$."

(5) Beginning of section 5: should be "section", not "chapter".

(6) Line 2 of Algorithm 1: Typo, should be $\max(48, 2\mathtt{r})$ (according to Lemma 4.10).

(7) Line 5 of Algorithm 3: "for" should be **for**, and it is missing **do**. This step just initializes `divisors` to be an array of 1s.

(8) Line 5 of Algorithm 4: should be $(\mathtt{a},\mathtt{b},\mathtt{c},r)$.

(9) Theorem 5.2: To be more precise, this theorem proves the correctness of FIND_R_ARITHMETIC.

(10) Proof of Theorem 5.2: The middle paragraph is a bit muddled. Here is a revised version that hopefully makes the logic clear.

We have from Lemma 3.4 that for every (a,b,c) and every $k \in \mathbb{Z}_{>0}$ with $k \in (\mathbb{Z}/2m\mathbb{Z})^\times$, if $\kappa(a,b,c;k) > 0$ then
$$c < \frac{k_c}{|k_a/a + k_b/b - 1|} \leq \frac{k}{|k_a/a + k_b/b - 1|} = \frac{kab}{|k_ab + k_ba - ab|}.$$
Suppose (a,b,c) is r-arithmetic, and consider the set of primes $q < c/2$ with $q \nmid m$. As in the proof of Lemma 4.3, these primes are distinct in $(\mathbb{Z}/2m\mathbb{Z})^\times/H$, so there at most $r-1$ of them for which $\kappa(a,b,c;q) < 0$; so among any r of them, there is at least one with $\kappa(a,b,c;q) > 0$. Putting

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these observations together, we find that in any set of r primes with $q \nmid ab$, there is a prime q in the set where at least one of the following holds:

$$\text{either } q \mid c \text{ or } c < 2q \text{ or } c < \left\lceil \frac{qab}{|q_a b + q_b a - ab|} \right\rceil.$$

To finish, suppose (a, b, c) is r-arithmetic, $c > 2{*}\texttt{maxNDP}$, and $c > \texttt{bound}$, where `bound` is in the keyset of `boundToPrimes`. Let

$$B = \{q : q \in \texttt{boundToPrimes}[\texttt{bound}'] \text{ and } \texttt{bound}' \leq \texttt{bound}\}.$$

Then there exist at most $r - 1$ primes $q \in B$ that do not divide c. The algorithm partitions B into r sets, and lets each `divisor` in `divisors` be the product of primes in one such set. Hence, c must be a multiple of at least one `divisor` in `divisors`. Therefore, the algorithm checks all possible r-arithmetic triples (a, b, c).

References

[1] Steve Nugent and John Voight, *On the arithmetic dimension of triangle groups*, Math. Comp. **86** (2017), no. 306, 1979–2004.