QUATERNION ORDERS OF CLASS NUMBER ONE (HW #7)

MATH 727

In this lab, we determine all definite quaternion orders of class number 1 over $\mathbb{Q}$.

(1) First, let $\mathcal{O}$ be a maximal order in a definite quaternion algebra B over $\mathbb{Q}$. Use the Eichler mass formula to get a bound on the discriminant D of B.

(2) Use Magma to find all maximal orders of class number 1. For example:

```
> B := QuaternionAlgebra(3*5*7);
> Discriminant(B);
> IsDefinite(B);
> O := MaximalOrder(B);
> Basis(O);
> H := RightIdealClasses(O);
> #H;
> H;
```

(3) Now let Λ be any order in a definite quaternion order and suppose $\Lambda \subset \mathcal{O}$. Show that $\#\operatorname{Cl}\Lambda \geq \#\operatorname{Cl}\mathcal{O}$. *[Hint: If it helps, think adelically!]*

(4) Let $\mathcal{O}$ be an *Eichler order*, an order such that $\mathcal{O}_p$ is principal for all ramified primes p and such that

$$\mathcal{O}_p \cong \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) : p^f \mid c \right\} \subset M_2(\mathbb{Z}_p)$$

(with $f \in \mathbb{Z}_{\geq 0}$, equal to 0 for almost all p) if p is split. We let $N = \prod_p p^f$ be the *level* of $\mathcal{O}$.

Making the obvious generalization to orders over a number field F, a variant of the Eichler mass formula for definite Eichler orders $\mathcal{O}$ of level $\mathfrak{N}$ reads:

$$\sum_{[J]\in\mathcal{O}} \frac{1}{w(J)} = 2^{1-n}|\zeta_F(-1)|h_F\Phi(\mathfrak{D})\Psi(\mathfrak{N})$$

where $\Psi(\mathfrak{N}) = N(\mathfrak{N})\prod_{\mathfrak{p}\mid\mathfrak{N}}(1 + 1/N\mathfrak{p})$.

Use this formula to find all definite Eichler orders with class number one. *[Hint: The answer is* 12*. At least I wrote a paper saying so. Vignéras says* 10*, but I think she only computes squarefree level (p. 153). Brzezinski quotes Vignéras as saying* 10*, but then I think he writes down the other* 2 *orders and just does not recognize that they are Eichler...]*

For example:

```
> // Eichler order of level 9 in algebra of discriminant 2
> O := QuaternionOrder(2,9);
> #RightIdealClasses(O);
```

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(5) Now the hard final step: use the previous exercise to find all quaternion orders with class number one.

For example:

```
> B<i,j,k> := QuaternionAlgebra<Rationals() | -1, -1>;
> O := QuaternionOrder([1,i,j,i*j]); // Not Eichler
> #RightIdealClasses(O);
```

Recall this is the order with reduced norm given by the sum of four squares!

(6) Let Λ be an order and suppose $\Lambda \subset \mathcal{O}$ where $\mathcal{O}$ is maximal and $h(\mathcal{O}) = \# \operatorname{Cl}(\mathcal{O}) = 1$. Then there exists $e \in \mathbb{Z}_{>0}$ such that $e\Lambda \subset \mathcal{O}$. Show that each class in $\operatorname{Cl}\Lambda$ has a representative I such that $e\mathcal{O} \subset I \subset \mathcal{O}$. *[Hint: For any such I, consider $I\mathcal{O} = x\mathcal{O}$; conclude that $e\mathcal{O} \subset x^{-1}I \subset \mathcal{O}$.]*

(7) What does the previous exercise tell you about the zeta function of a general order (over $\mathbb{Q}$)? Do you conjecture a version of Eichler's mass formula for them?