In this lab, we determine all definite quaternion orders of class number 1 over \( \mathbb{Q} \).

1. First, let \( \mathcal{O} \) be a maximal order in a definite quaternion algebra \( B \) over \( \mathbb{Q} \). Use the Eichler mass formula to get a bound on the discriminant \( D \) of \( B \).

2. Use Magma to find all maximal orders of class number 1. For example:
   ```
   > B := QuaternionAlgebra(3*5*7);
   > Discriminant(B);
   > IsDefinite(B);
   > O := MaximalOrder(B);
   > Basis(O);
   > H := RightIdealClasses(O);
   > #H;
   > H;
   ```

3. Now let \( \Lambda \) be any order in a definite quaternion order and suppose \( \Lambda \subset \mathcal{O} \). Show that \( \# Cl \Lambda \geq \# Cl \mathcal{O} \). [Hint: If it helps, think adelically!]

4. Let \( \mathcal{O} \) be an Eichler order, an order such that \( \mathcal{O}_p \) is principal for all ramified primes \( p \) and such that

   \[
   \mathcal{O}_p \cong \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) : p^f | c \right\} \subseteq M_2(\mathbb{Z}_p)
   \]

   (with \( f \in \mathbb{Z}_{\geq 0} \), equal to 0 for almost all \( p \)) if \( p \) is split. We let \( N = \prod_p p^f \) be the level of \( \mathcal{O} \).

   Making the obvious generalization to orders over a number field \( F \), a variant of the Eichler mass formula for definite Eichler orders \( \mathcal{O} \) of level \( \mathfrak{N} \) reads:

   \[
   \sum_{[J] \in \mathcal{O}} \frac{1}{w(J)} = 2^{1-n} |\zeta_F(-1)| h_F \Phi(\mathfrak{O}) \Psi(\mathfrak{N})
   \]

   where \( \Psi(\mathfrak{N}) = N(\mathfrak{N}) \prod_{p | \mathfrak{N}} (1 + 1/Np) \).

   Use this formula to find all definite Eichler orders with class number one. [Hint: The answer is 12. At least I wrote a paper saying so. Vignéras says 10, but I think she only computes squarefree level (p. 153). Brzezinski quotes Vignéras as saying 10, but then I think he writes down the other 2 orders and just does not recognize that they are Eichler...]

   For example:
   ```
   > // Eichler order of level 9 in algebra of discriminant 2
   > O := QuaternionOrder(2,9);
   > #RightIdealClasses(O);
   ```

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(5) Now the hard final step: use the previous exercise to find all quaternion orders with class number one.

For example:

```maple
> B<i,j,k> := QuaternionAlgebra<Rationals() | -1, -1>;
> O := QuaternionOrder([1,i,j,i*j]); // Not Eichler
> #RightIdealClasses(O);
```

Recall this is the order with reduced norm given by the sum of four squares!

(6) Let \( \Lambda \) be an order and suppose \( \Lambda \subset O \) where \( O \) is maximal and \( h(O) = \# \text{Cl}(O) = 1 \). Then there exists \( e \in \mathbb{Z}_{>0} \) such that \( e\Lambda \subset O \). Show that each class in \( \text{Cl}\Lambda \) has a representative \( I \) such that \( eO \subset I \subset O \). [Hint: For any such \( I \), consider \( IO = xO; \) conclude that \( eO \subset x^{-1}I \subset O \).]

(7) What does the previous exercise tell you about the zeta function of a general order (over \( \mathbb{Q} \))? Do you conjecture a version of Eichler’s mass formula for them?