DEFORMATIONS OF QUOTIENT SINGULARITIES
(BY CYCLIC GROUPS)

OSWALD RIEMENSCHNEIDER

This work owes its origins to the desire to construct examples of \( \text{Zusammenblasen} \) of deformations of the resolutions of a rational singularity to deformations of the singularities themselves ([1]). We thereby begin with limited rational singularities, those for which the dual graph of the exceptional divisor system of a minimal resolution is assumed for simplicity to be normal. These are the well-known two-dimensional quotient singularities which result from the action of a cyclic group on \( \mathbb{C}^2 \).

These singularities can be described after Brieskorn [1] as follows: Given natural numbers \( n \) and \( q \) with \( 0 < q < n \), \( \gcd(n, q) = 1 \), \( \zeta_n \) a primitive \( n \)th root of unity, the map \( \phi_{n,q} : \mathbb{C}^2 \to \mathbb{C}^2 \) acting via \( \phi_{n,q}(u,v) = (\zeta_n u, \zeta_q^n v) \) generates a group of automorphisms \( G_{n,q} \) on \( \mathbb{C}^2 \) which is cyclic and has a single fixed point \( 0 \); let \( X_{n,q} \) be the analytic space of germs of \( \mathbb{C}^2/G_{n,q} \) in a neighborhood of the origin. A quotient singularity is analytically isomorphic to one such \( X_{n,q} \) and furthermore \( X_{n,q} \cong X_{n',q'} \) iff \( n = n' \) and \( q = q' \) or \( qq' \equiv 1 \mod n \). One can obtain the dual graph of the exceptional divisor system in the minimal resolution \( \tilde{X}_{n,q} \) of the singularity of \( X_{n,q} \) by means of the Hirzebruch-Jung continued fraction: With

\[
\frac{n}{q} = b_1 - \frac{1}{b_2 - \frac{1}{\ldots - \frac{1}{b_r}}} \quad \text{for } b_i \in \mathbb{Z}, b_i \geq 2, i = 1, \ldots, r,
\]

one obtains the dual graph

\[
-b_1 \quad -b_2 \quad -b_{r-1} \quad -b_r \quad \bullet \quad \ldots \quad \bullet \quad (\bullet \cong \mathbb{P}^1).
\]

In order to find deformations of \( X_{n,q} \), one must calculate as the first \( \text{zugehörige} \) analytic \( \text{Stellenalgebra} \) \( A_{n,q} \). This can be done by the above description by means of invariant theory \( \text{durchgeführt werden} \) (§§1,2). It turns out that the equations for \( X_{n,q} \) can be read off directly from the Hirzebruch-Jung continued fraction expansion for \( n/(n-q) \). Since there are simple relations between the fractions for \( n/q \) and \( n/(n-q) \) (§3), one can also read the equations directly from the dual graph of the resolution. In particular, the following are equivalent:

(a) \( X_{n,q} \) is a \text{vollständiger Durchschnitt};
(b) \( q = n - 1 \);
(c) \( X_{n,q} \) is the singular hypersurface \( x_1x_3 - x_2^n = 0 \);

This is a coarse translation of §§1-4 of the German text Oswald Riemenschneider, \textit{Deformationen von Quotientensingularitäten (nach zyklischen Gruppen)}, Math. Ann. 209 (1974), 211–248, by John Voight (jvoight@math.berkeley.edu).
(d) The dual graph of the resolution $\tilde{X}_{n,q}$ of $X_{n,q}$ is

$\phantom{-2} \phantom{-2} \phantom{-2} \phantom{-2} \phantom{\ldots} \phantom{-2} \phantom{-2}$

(with $n-1$ components).

Generally the singularities are not $X_{n,q}$ vollständiger Durchschnitt. Nevertheless, one may succeed in determining all relations between the defining equations (§4) and construct an interesting family of singularities with special fiber $X_{n,q}$ (§5). The singularities of the neighboring fibers of $X_{n,q}$ in this family are all of type $X_{n',q'}$; we are able to indicate complete graphs of the occurring singularities. The family always possesses a single parameter underlying the family with singular generic fiber. This implies that no singularity $X_{n,q}$ is rigid.

The resolution $\tilde{X}_{n,q}$ of $X_{n,q}$ likewise possesses a canonical deformation (§6). We can show that from Zusammenblasen of this deformation (essentially) the above deformation of $X_{n,q}$ results. Thus the variation of the dual graphs described in §5 is explained in the similar behavior of the exceptional curves with deformation of the surrounding variety. We can therefore isolate the following occurring phenomena:

1. Irreducible components disappear; and
2. Two transverse cutting components with self-intersection numbers $-b_1$ and $-b_2$ combine to only one singular rational curve with self-intersection number $-(b_1 + b_2 - 2)$.

Finally, we indicate (§7) a formula for the dimension of the vector space of all infinitesimal deformations of $X_{n,q}$, the Mumfords formula [9] for $X_{n,1}$, which is the cone over the rational curve of degree $n$ in $\mathbb{P}^n$. From this, it follows that the deformation indicated in (§5) is not versal in the case that the embedding dimension of $A_{n,q}$ is $e \geq 5$. For $e = 4$ (as for $e = 3$) it is correct however with the versal deformation over it. We can construct the versal family for $e = 5$, which generalizes the result by Pinkham [12] for $X_{4,1}$. Contrary to [12], where one gets along with simple homogeneous examination, our proof requires a more extensive calculation. Although in the same way one can treat the higher embedding dimension, because of the extent of computation in its implementation we will restrict ourselves to some further remarks on the case $e = 6$. One can find results for the cones $X_{n,1}$ in Pinkham.

A set of problems that could not be resolved is then presented. Besides the results obtained, one must permit the conjecture that appropriate holding back relationship with the deformation of any rational singularity prevail. We arranged a list of the open questions and conjectures in the last paragraph.

We regard somewhat more general analytic Stellenalgebren over any complete assessed field $k$. The chosen designations follow as in [4]: e.g. $K_n = k\langle x_1, \ldots, x_n \rangle$ is the algebra of convergent power series in $n$ variables $x_1, \ldots, x_n$ with coefficients in $k$.

1. INVARIANTS

Let $k$ be an algebraically closed field; let $\zeta_n$ be a primitive $n$th root of unity with $n \not\equiv 0 \pmod{\text{char } k}$. Also let $0 < q < n$, $\gcd(n, q) = 1$. Then

$\phi = \phi_{n,q} : \phi(u) = \zeta_n u, \quad \phi(v) = \zeta_n^q v$

generates a group action $G = G_{n,q}$ on $K_2 = k\langle u, v \rangle$. 
We want to determine in this paragraph the invariants of $\phi$. Since $G$ is a finite group, one can do as follows after a remark of Noether [10]: One forms the polynomials
\[
\prod_{\psi \in G} (Y - \psi(u)), \quad \prod_{\psi \in G} (Y - \psi(v))
\]
and after multiplying out one has in the coefficients certain invariants. In our case, we have
\[
\prod_{\psi \in G} (Y - \psi(u)) = n^{-1} \prod_{\nu=0}^{n-1} (Y - \zeta_n^\nu u) = Y^n - u^n
\]
and similarly
\[
\prod_{\psi \in G} (Y - \psi(v)) = Y^n - v^n.
\]
This gives the invariants $u^n$ and $v^n$. Following Noether, one can get a full system of invariants by adding the elements $\mu(u^i v^j), 0 \leq i < n, 0 \leq j < n$, where $\mu : K_2 \to K_2^G$ is the well-known center of mass (viz. [4], p. 158): It is now
\[
\mu(u^i v^j) = \frac{1}{n} \sum_{\psi \in G} \psi(u^i v^j) = \left( \frac{1}{n} \sum_{\nu=0}^{n-1} (\zeta_n^{i+qj})^\nu \right) u^i v^j
\]
and we have thus
\[
\text{Theorem 1. The invariant $k$-algebra } S_G^2 \text{ where } S = k[u, v] \text{ is generated by those monomials } u^i v^j \text{ with } i + qj \equiv 0 \pmod{n}, 0 \leq i \leq n, 0 \leq j \leq n, (i, j) \neq (n, n).
\]
Theorem 1 however does not in general give a minimal generating system, as the following example shows: Let $n$ be arbitrary and $q = n - 1$. Then one has by Theorem 1 the invariants
\[
1, u^n, v^n, uv, u^2 v^2, \ldots, u^{n-1} v^{n-1}
\]
of which $u^2 v^2, \ldots, u^{n-1} v^{n-1}$ are obviously superfluous.
We can however succeed with the aid of the Hirzebruch-Jung algorithm which refines the above generating set to a minimum. We set (viz. [6, 7]):
\[
i_1 = n, \quad i_2 = n - q.
\]
Then we can find a natural number $a_2 \geq 2$ such that
\[
i_1 = a_2 i_2 - i_3, \quad 0 \leq i_3 < i_2.
\]
Accordingly, one may continue this process and we obtain the development of a continued fraction
\[
i_1 = a_2 i_2 - i_3
\]
\[
i_2 = a_3 i_3 - i_4
\]
\[
\vdots
\]
\[
i_{e-2} = a_{e-1} i_{e-1} - i_e
\]
with \( a_2 \geq 2, \ldots, a_{e-1} \geq 2 \), \( i_1 = n > i_2 = n - q > i_3 > \cdots > i_{e-1} = 1 > i_e = 0 \), \( e \geq 3 \). It implies then
\[
\frac{n}{n-q} = a_2 - \frac{1}{a_3 - \frac{1}{\cdots - a_{e-1}} = \lfloor a_2; a_3, \ldots, a_{e-1} \rfloor.
\]

Contrariwise given \( a_2 \geq 2, \ldots, a_{e-1} \geq 2 \), the \( i_e \) can be calculated inductively via
\[
(2) \quad i_1 = n, \quad i_2 = n - q, \quad i_{e+1} = a_e i_e - i_{e-1}, \quad e = 2, \ldots, e - 1.
\]
We can thus inductively define the numbers
\[
(3) \quad j_1 = 0, \quad j_2 = 1, \quad j_{e+1} = a_e j_e - j_{e-1},
\]
\[
(4) \quad k_1 = 1, \quad k_2 = 1, \quad k_{e+1} = a_e k_e - k_{e-1}.
\]
Because \( a_e \geq 2 \) it follows easily by induction that
\[
(5) \quad j_1 < j_2 < \cdots < j_e
\]
\[
(5) \quad k_1 \leq k_2 \leq \cdots \leq k_e.
\]

Additional one can obtain the following equations without much pain:
\[
\begin{align*}
    i_e + q j_e &= n k_e, & \varepsilon &= 1, \ldots, e, \\
    j_{e+1} k_e - j_e k_{e+1} &= n, \\
    k_{e+1} k_e - k_e k_{e+1} &= q,
\end{align*}
\]

and thus in particular \( \gcd(j_e, k_e) = 1, \varepsilon = 1, \ldots, e \). From the theory of continued fractions (viz. e.g. [11]) that \( j_{e+1} \) is the numerator of the convergent \( \lfloor a_2; a_3, \ldots, a_{e+1} \rfloor \). This implies that \( j_e = n \) and because \( i_e = 0 \) that \( k_e = q \).

**Theorem 2.** The \( k \)-algebra \( S^G \) is generated by 1 and the elements \( u^i v^j, \varepsilon = 1, \ldots, e \).

Before we prove Theorem 2, we prove the following

**Proposition 1.** Let \((i, j, k)\) satisfy \( i + q = nk \) with \( 0 \leq i < n \) and \( 0 < j \leq n \). In order for
\[
\frac{j}{k} > \frac{j_e}{k_e}
\]
for some \( \varepsilon \) with \( 1 \leq \varepsilon \leq e \), it is necessary that \( \varepsilon < e \) and \( j > j_{e+1} \).

**Proof.** Because \( i + q = nk \) we have
\[
\frac{j}{k} = \frac{n}{q} - \frac{i}{qk} \leq \frac{n}{qk} = \frac{j_e}{k_e}
\]
and therefore \( \varepsilon < e \). Since the sequence \( j_e \) is monotonically increasing with respect to \( \varepsilon \), we can assume that \( \varepsilon < e \) is maximal with respect to \( j/k > j_e/k_e \). Now assume that \( j < j_{e+1} \). Then \( k \leq k_{e+1} \) and using (5) we have
\[
\frac{j}{k} = \frac{j k e - j k e}{k k e} + \frac{j e}{k e} \geq \frac{1}{k e k_{e+1}} + \frac{j e}{k e} = \frac{j e+1}{k e+1}.
\]
Since \( \varepsilon \) is maximal we must have \( j k e+1 = k j e+1 \) and because of the relative primality of \( j_{e+1} \) and \( k_{e+1} \) we have \( j_{e+1} \mid j \) and so \( j \geq j_{e+1} \). Contradiction!
Proof of Theorem 2. From \( i_\varepsilon + q_j \varepsilon = nk_\varepsilon, \varepsilon = 1, \ldots, \varepsilon, 0 \leq i_\varepsilon \leq n, 0 \leq j_\varepsilon \leq n, \) the monomial \( u^i \varepsilon v^j \) is contained in the generating set of Theorem 1. Therefore it suffices to show that each monomial \( u^i \varepsilon v^j \) with \( i + q_j \varepsilon \equiv 0 \pmod{n} \) can be represented as a polynomial in the \( u^i \varepsilon v^j \). But this follows easily from the following statement:

If \( j_\varepsilon < j < j_{\varepsilon+1}, \varepsilon < \varepsilon, \) and \( i \) the unique solution to \( i + q_j \varepsilon \equiv 0 \pmod{n} \) (\(^\ast\) with \( 0 \leq i < n \), then \( i \geq i_\varepsilon \) (and thus \( i > i_\varepsilon \) since \( j \neq j_\varepsilon \)). Put differently: \( j_{\varepsilon + 1} \) is the smallest \( j > j_\varepsilon \) such that the membership \( i \) is smaller than \( i_\varepsilon \).

In order to prove (\(^\ast\)), one multiplies the equation \( i + q_j \varepsilon = nk_\varepsilon \) by \( k_\varepsilon \) and the equation \( i_\varepsilon + q_j \varepsilon = nk_\varepsilon \) by \( k \). After subtracting one obtains

\[
i k_\varepsilon - k i_\varepsilon + q(j k_\varepsilon - k j_\varepsilon) = 0.
\]

Now \( k_\varepsilon \leq k \) because \( j_\varepsilon < j \) and \( j k_\varepsilon - k j_\varepsilon \leq 0 \) because of the lemma proven above. Therefore this is

\[
0 \leq q(j k_\varepsilon - j k_\varepsilon) = ik_\varepsilon - ki_\varepsilon \leq k(i - i_\varepsilon)
\]

and thus \( i \geq i_\varepsilon \).

We have calculated thus far only the invariant polynomial algebra \( S_G^n \). But according to [4, Theorem III, 3.3] this thereby determines in addition the analytic algebra \( K^G_2 \): Let \( K_\varepsilon = k(x_1, \ldots, x_\varepsilon) \) and \( \alpha : K_\varepsilon \rightarrow K_2 \) via \( \alpha(x_\varepsilon) = u^i \varepsilon v^j, \varepsilon = 1, \ldots, \varepsilon \) which defines a \( k \)-algebra homomorphism; then

\[
K^G_2 \simeq K_\varepsilon / \ker \alpha.
\]

In the next section we will determine \( \ker \alpha \). As a consequence we know that \( e \) is the embedding dimension of \( K^G_2 \). This follows directly from

Corollary. The set indicated by the theorem is minimal.

2. Equations

Let \( A_{n,q} = K^G_{2,q} \) and \( X_{n,q} \) the corresponding germ of the analytic space. After the remark (6) \( X_{n,q} \) is an analytic Unterraumkeim of \( (k^e, 0) \). In order to find the defining equations for \( X_{n,q} \), we must calculate a generating set of \( \ker \alpha \). We let \( \lfloor n/(n - q) \rfloor \) is the continued fraction expansion of \( n/(n - q) \) as in \( \S 1 \):

\[
p_{k\varepsilon} = \left\{ \begin{array}{ll}
a_{k+1}, & \delta + 1 = \varepsilon - 1, \\
a_{k+1} - 1, a_{k+2}, \ldots, a_{k+2} - 2, a_{k+1} - 1, & \delta + 1 < \varepsilon - 1,
\end{array} \right.
\]

(1 \leq \delta, \varepsilon \leq e), and

\[
g_{k\varepsilon} = x_k x_\varepsilon - p_{k\varepsilon}, \quad 2 \leq \delta + 1 \leq \varepsilon - 1 \leq e - 1.
\]

Furthermore, let \( a = a_{n,q} \) be the ideal generated by \( g_{k\varepsilon} \) in \( K_\varepsilon \). We will show that \( a_{n,q} = \ker \alpha \). We begin the proof with

Lemma 1. \( a \subset \ker \alpha \).

Proof. This follows from (2) and (3): \( i_\delta + i_{\delta + 2} = a_{\delta + 1} i_{\delta + 1} \) and \( j_\delta + j_{\delta + 2} = a_{\delta + 1} j_{\delta + 1} \), and from this the result arises directly from induction on \( \varepsilon \) for \( \delta + 1 < \varepsilon - 1 \):

\[
i_\delta + i_\varepsilon = (a_{\delta + 1} - 1)i_{\delta + 1} + (a_{\delta + 2} - 2)i_{\delta + 2} + \cdots + (a_{\varepsilon - 2} - 2)i_{\varepsilon - 2} + (a_{\varepsilon - 1} - 1)i_{\varepsilon - 1}
\]

and similarly for \( j_\delta + j_\varepsilon \). This immediately implies \( \alpha(g_{k\varepsilon}) = 0 \) as claimed. \( \square \)
For the reverse inclusion we need a further proposition. We define
\[ Z^2_+ = \{ (\beta, \gamma) \in \mathbb{Z} \times \mathbb{Z} : \beta \geq 0, \gamma \geq 0, (\beta, \gamma) \neq (0,0) \}. \]

**Proposition 2.** For fixed \((\beta, \gamma) \in Z^2_+\), consider the system of equations
\[
\begin{align*}
\lambda^{(1)}_e i_e + \lambda^{(2)}_e i_{e+1} &= \beta, \\
\lambda^{(1)}_e j_e + \lambda^{(2)}_e j_{e+1} &= \gamma
\end{align*}
\]
\((\ast_e)\)

The system of equations \((\ast_\delta)\) has a solution \((\lambda^{(1)}_\delta, \lambda^{(2)}_\delta) \in Z^2_+\), and
(a) in the case \(\lambda^{(1)}_\delta = 0, \delta + 1 \leq e - 1\) then \((\ast_{\delta+1})\) has a solution in \(Z^2_+\), namely
\[ (\lambda^{(1)}_{\delta+1}, \lambda^{(2)}_{\delta+1}) = (\lambda^{(2)}_\delta, 0), \]
(b) in the case \(\lambda^{(2)}_\delta = 0, \delta - 1 \geq 1\) then \((\ast_{\delta-1})\) has a solution in \(Z^2_+\), namely
\[ (\lambda^{(1)}_{\delta-1}, \lambda^{(2)}_{\delta-1}) = (0, \lambda^{(1)}_\delta), \]
(c) in all other cases the system \((\ast_e)\), \(\varepsilon \neq \delta\) has no solution in \(Z^2_+\).

**Proof.** From (5) \(i_e j_{e+1} - i_{e+1} j_e = n\). Thus \((\ast_e)\) possesses a solution \((\lambda^{(1)}_e, \lambda^{(2)}_e) \in \mathbb{Q}^2\), namely
\[ \lambda^{(1)}_e = \frac{1}{n} (\beta j_e - \gamma i_e), \quad \lambda^{(2)}_e = \frac{1}{n} (\beta j_e + \gamma i_e). \]
Because of (1), (4) and \((\beta, \gamma) \in Z^2_+\) we have \(\lambda^{(1)}_e < \lambda^{(1)}_{e+1}, \lambda^{(2)}_e > \lambda^{(2)}_{e+1}\).

Now let \(\delta\) be maximally refined, so that \((\lambda^{(1)}_\delta, \lambda^{(2)}_\delta) \in Z^2_+\). Then the following cases occur:
1) \(\lambda^{(2)}_\delta = 0, \delta > 1\). Then \(\lambda^{(1)}_\delta > 0, \lambda^{(1)}_{\delta-1} = -\lambda^{(2)}_\delta = 0, \lambda^{(2)}_{\delta-1} = \lambda^{(1)}_\delta > 0\) and \(\lambda^{(1)}_e < \lambda^{(1)}_{e+1} = 0\) for all \(\varepsilon < \delta - 1\). This gives the only solution to \((\ast_{\delta-1})\) in \(Z^2_+\).
2) \(\lambda^{(2)}_\delta > 0, \delta > 1\). Then \(\lambda^{(1)}_\delta \leq \lambda^{(1)}_{\delta-1} = -\lambda^{(2)}_\delta < 0\) for all \(\varepsilon < \delta - 1, i.e.\) no further solution exists in \(Z^2_+\).
3) \(\delta = 1\). Then there is likewise no further solution in \(Z^2_+\). \(\square\)

**Theorem 3.** \(\ker \alpha = a_{n,q}\).

**Proof.** Because of Lemma 1 we need only show \(\ker \alpha \subset a_{n,q}\). Let
\[ f = \sum_{\nu_1, \ldots, \nu_e = 0}^\infty a_{\nu_1, \ldots, \nu_e} x_1^{\nu_1} \ldots x_e^{\nu_e} \in \ker \alpha. \]
We thus define \(I_{\beta, \gamma} = \{ (\nu_1, \ldots, \nu_e) : \alpha(x_1^{\nu_1} \ldots x_e^{\nu_e}) = u^0 v^\gamma \} \) for \((\beta, \gamma) \in Z^2_+\) and find
\[ f = f_0 + \sum f_{\beta, \gamma} \]
where
\[ f_0 = a_{0,0} \quad \text{and} \quad f_{\beta, \gamma} = \sum_{(\nu_1, \ldots, \nu_e) \in I_{\beta, \gamma}} a_{\nu_1, \ldots, \nu_e} x_1^{\nu_1} x_e^{\nu_e}. \]
Obviously one also has \(f_0, f_{\beta, \gamma} \in \ker \alpha\), from which the following identity follows immediately:
\[ a_{0,0} = 0, \quad \sum_{(\nu_1, \ldots, \nu_e) \in I_{\beta, \gamma}} a_{\nu_1, \ldots, \nu_e} x_1^{\nu_1} \ldots x_e^{\nu_e} = 0, \quad (\beta, \gamma) \in Z^2_+. \]
It is enough now to show that $f_{\beta,\gamma} \in a_{n,q}$. Then situated $f = \sum f_{\beta,\gamma}$ for the produced ideal of $a_{n,q}$ in the completion $k\{x_1, \ldots, x_e\}$ of $k(x_1, \ldots, x_e)$ by the Krull Lemma ([4], p. 46) in $a_{n,q}$ itself.

Let $(\nu_1, \ldots, \nu_e) \in I_{\beta,\gamma}$. Then one can find another number $(\tilde{\nu}_1, \ldots, \tilde{\nu}_e) \in I_{\beta,\gamma}$, with at most two elements $\tilde{\nu}_e, \tilde{\nu}_{e+1}$ different from zero, so that
\[
x_1^{\nu_1} \ldots x_e^{\nu_e} \equiv x_1^{\tilde{\nu}_1} \ldots x_e^{\tilde{\nu}_e} \pmod{a_{n,q}}
\]
(one proves this without great difficulty e.g. by induction on $e$). So one has
\[
f_{\beta,\gamma} \equiv \tilde{f}_{\beta,\gamma} = \sum_{(\nu_1, \ldots, \nu_e) \in I_{\beta,\gamma}} a_{\nu_1, \ldots, \nu_e} x_1^{\tilde{\nu}_1} \ldots x_e^{\tilde{\nu}_e} \pmod{a_{n,q}}
\]
\[
\sum_{(\nu_1, \ldots, \nu_e) \in I_{\beta,\gamma}} a_{\nu_1, \ldots, \nu_e} = 0, \quad \alpha(x_1^{\nu_1} \ldots x_e^{\nu_e}) = u^\beta v^\gamma.
\]
From Lemma 2 it now follows that for $(\nu_1, \ldots, \nu_e), (\mu_1, \ldots, \mu_e) \in I_{\beta,\gamma}$ immediately
\[
x_1^{\nu_1} \ldots x_e^{\nu_e} = x_1^{\mu_1} \ldots x_e^{\mu_e}
\]
and thus for a fixed $(\mu_1, \ldots, \mu_e) \in I_{\beta,\gamma}$:
\[
\tilde{f}_{\beta,\gamma} = \left( \sum_{(\nu_1, \ldots, \nu_e) \in I_{\beta,\gamma}} a_{\nu_1, \ldots, \nu_e} \right) x_1^{\mu_1} \ldots x_e^{\mu_e} = 0
\]
i.e.
\[
f_{\beta,\gamma} \in a_{n,q}.
\]

Therefore $A_{n,q} = K^G_{n,q} \simeq K/I_{\beta,\gamma}$ for $K_{n,q}$ is an invariant subalgebra of the regular algebra $K_2$ implies it is normal, so in particular $a_{n,q}$ is a prime ideal. Additionally $a_{n,q} \subset m^2_\varepsilon$ if $m_\varepsilon$ defines the maximal ideal of $k(x_1, \ldots, x_e)$. This implies for the embedding dimension $\text{embdim}(A_{n,q})$ the formula

**Lemma 2.** $\text{embdim}(A_{n,q}) = e$.

We want finally the minimal number $\text{numgen}(a_{n,q})$ of generators for the ideal $a_{n,q}$. If we let
\[
e_1 := \frac{1}{2}(e - 1)(e - 2)
\]
Then therefore

**Theorem 4.** $\text{numgen}(a_{n,q}) = e_1$.

**Proof.** We have $\text{numgen}(a_{n,q}) = \text{dim}_k a_{n,q}/m_a_{n,q}$. It is enough from this to show that the residue class of the $e_1$ generators $g_{3\varepsilon}$, $2 \leq \delta + 1 \leq e - 1 \leq e - 1$, in $a_{n,q}/m_a_{n,q}$ are linearly independent over $k$. Suppose therefore there are $c_{3\varepsilon} \in k$ and
\[
\sum c_{3\varepsilon} g_{3\varepsilon} \in m_a_{n,q}.
\]
Then $g_{1\varepsilon} = x_1 x_\varepsilon - p_{1\varepsilon}$ and $a_{n,q} \subset m^2_\varepsilon$ give therefore a polynomial $P(x_2, \ldots, x_e)$ with
\[
\sum_{\varepsilon=3} c_{1\varepsilon} x_1 x_\varepsilon + P(x_2, \ldots, x_e) \in m^3_\varepsilon,
\]
from which \( c_{1\varepsilon} = 0, \varepsilon = 3, \ldots, e \) is obtained. One proves inductively in the same way that \( c_{\delta\varepsilon} = 0 \) for all \( \delta, \varepsilon \) with \( 2 \leq \delta + 1 \leq \varepsilon - 1 \leq e - 1 \).

**Conclusion.** \( A_{n,q} \) is a vollständiger Durchschnitt exactly when \( q = n - 1 \). In this case \( A_{n,q} \) is the singular hypersurface \( k(x_1, x_2, x_3)/(x_1 x_3 - x_2^2) \).

**Proof.** \( A_{n,q} \) is vollständiger Durchschnitt \( \iff e = \text{embdim} A_{n,q} = \dim A_{n,q} + \numgen a_{n,q} = 2 + e_1 \iff e = 3 \iff n - q = 1 \iff a_2 = n. \)

3. Resolution of Singularities and the Continued Fraction Expansion for \( n/q \) and \( n/(n - q) \)

The algebra \( A_{n,q} \) is generated by the elements \( u^i v^j \in k(u, v), \varepsilon = 1, \ldots, e; \) in other words,

\[
A_{n,q} = k\langle u^i v^j \rangle.
\]

Of these elements, \( u^n, u^{n-q} v, v^n \) always occur. We set therefore

\[
B_{n,q} = k\langle u^n, u^{n-q} v, v^n \rangle.
\]

The analytic monomorphism \( B_{n,q} \hookrightarrow A_{n,q} \) is obtained. Since for the quotient fields \( Q(B_{n,q}) = Q(A_{n,q}) \) and \( A_{n,q} \) is normal, \( A_{n,q} \) is the normalization of \( B_{n,q} \). In order to obtain the resolution of singularities from \( X_{n,q} \), it is enough to singularities of \( B_{n,q} \) belonging to the space of germs \( Y_{n,q} \) up less. Now since

\[
B_{n,q} \simeq k(x_1, x_2, x_e)/(x_1^{n-q} x_e - x_2^n)
\]

and with the well-known resolution of \( Y_{n,q} \) by Hirzebruch [6] (viz. also Laufer [8]): One forms to start the continued fraction expansion

\[
\frac{n}{q} = [b_1; b_2, \ldots, b_r], \quad b_\varrho \geq 2, \quad \varrho = 1, \ldots, r,
\]

and thus one fastens \( r + 1 \) copies of \( k^2 \) with coordinates \( (u_\varrho, v_\varrho) \), \( \varrho = 0, \ldots, r \) successively in the following way:

\[
\begin{align*}
  u_1 &= \frac{1}{u_0}, & v_1 &= u_0^{b_1} v_0 \\
  v_2 &= \frac{1}{v_1}, & u_2 &= v_1^{b_2} u_1 \\
  & \vdots
\end{align*}
\]

(10)

In this way, one obtains a variety \( \tilde{X}_{n,q} \), which together with a suitable map \( \tilde{X}_{n,q} \xrightarrow{\pi} Y_{n,q} \) resolves the singularities of \( Y_{n,q} \). The exceptional divisor \( E = \pi^{-1}(0) \) is a union of \( r \) rational nonsingular curves \( E_\varrho \simeq \mathbb{P}_1(k) \), \( \varrho = 1, \ldots, r \), with the following intersection matrix:

\[
E_\varrho \cdot E_\sigma = \begin{cases} 
-b_\varrho, & \sigma = \varrho \\
1, & \sigma = \varrho + 1, \sigma - 1 \\
0, & \text{otherwise}.
\end{cases}
\]

(11)

The corresponding dual graph is then

\[
-b_1 \rightarrow -b_2 \rightarrow -b_{r-1} \rightarrow -b_r, \quad \bullet \rightarrow \mathbb{P}_1(k).
\]
The map \( \pi \) gives a map \( f \) of the normal factor rings \( X_{n,q} \to Y_{n,q} \). We will explicitly describe \( f \) in the following theorem which does not seem to have yet occurred so far in the literature.

**Theorem 5.** The functions \( f_\epsilon = u_0^{\epsilon} v_0^{k_\epsilon}, \quad \epsilon = 1, \ldots, e \) can be extended holomorphically to all of \( \tilde{X}_{n,q} \). The map \( f : \tilde{X}_{n,q} \to X_{n,q} \) is given by \( f = (f_1, \ldots, f_e) \).

The proof we leave to the reader (one can use induction over \( r \) and thereby the following observation; see also the proof sketch for Theorem 9).

The calculation of \( n \) and \( q \) of \( b_1, \ldots, b_e \) gives in reverse the numbers \( a_2, \ldots, a_{e-1} \) (i.e. from the dual graph of the resolution of the equations). We want to now show that there is also a direct way to do this calculation. Two important formulas result for the connection between \( b_\rho \) and the \( a_\epsilon \).

**Lemma 3.** We have

\[
\frac{n}{q} = [[b_1; b_2, \ldots, b_r]], \\
\frac{n_1}{q_1} = [[b_2; b_3, \ldots, b_r]], \\
\frac{n_1}{n_1 - q_1} = [[a_2; a_3, \ldots, a_{e-1}]].
\]

Therefore

\[
\frac{n}{n-q} = [[2; 2, \ldots, 2, (a_2 + 1), a_3, \ldots, a_{e-1}]]
\]

with \((b_1 - 2)\) twos at the start.

**Proof.** (By induction on \( b_1 \geq 2 \)) Let \( b_1 = 2 \). Then

\[
\frac{n}{q} = 2 - \frac{1}{n_1/q_1} = \frac{2n_1 - q_1}{n_1}
\]

and therefore

\[
\frac{n}{n-\frac{q}{q}} = \frac{2n_1 - q_1}{n_1} = 1 + \frac{n_1}{n_1 - q_1} = [[(a_2 + 1); a_3, \ldots, a_{e-1}]].
\]

Now the correctness of the statement is already proven for \( b_1 \geq 2 \), and

\[
\frac{n}{q} = [[(b_1 + 1); b_2, \ldots, b_r]].
\]

Thus

\[
\frac{n-\frac{q}{q}}{q} = [[b_1; b_2, \ldots, b_r]] = b_1 - \frac{1}{n_1/q_1}
\]

and so by the inductive hypothesis

\[
\frac{n-\frac{q}{q}}{n-2q} = [[2; 2, \ldots, 2, (a_2 + 1), \ldots, a_{e-1}]]
\]

with \((b_1 - 2)\) twos at the start. The statement now follows from

\[
\frac{n}{n-\frac{q}{q}} = 2 - \frac{1}{n-2q}
\]

\(\square\)
Lemma 3 supplies the following practical method for calculating the $a_\varepsilon$ from the $b_\varrho$. One orders $(b_1, \ldots, b_t)$ a system of points in the following manner:

\[
\begin{array}{c}
\times \times \ldots \times \\
(\text{b}_1 - 1) \text{ points} \\
\times \times \ldots \times \\
(\text{b}_2 - 1) \text{ points} \\
\times \times \ldots \times \\
(\text{b}_3 - 1) \text{ points} \\
\ldots
\end{array}
\]

Each system gives as its vertical columns of points an $a_\varepsilon$, and indeed $a_\varepsilon - 1$ is equal to the number of points.

Example 1. $(b_1, \ldots, b_5) = (5, 2, 2, 3, 2)$ gives the diagram

\[
\begin{array}{c}
\times \times \times \times \\
\times \\
\times \\
\times \\
\times \\
\end{array}
\]

and so $(a_2, \ldots, a_6) = (2, 2, 5, 3)$. One can easily verify (though with some considerable expense) that indeed

\[
[[5; 2, 2, 3, 2]] = \frac{47}{11}
\]

and

\[
\frac{47}{11 - 11} = [[2; 2, 5, 3]].
\]

The following statement follows immediately from Lemma 3:

**Lemma 4.** Let $\frac{n}{q} = [[b_1; b_2, \ldots, b_t]]$ and $\frac{n}{n - q} = [[a_2; a_3, \ldots, a_{e-1}]]$.

Then

(i) $\sum_{\varrho} (b_\varrho - 1) = \sum_{\varepsilon=2}^{e-1} (a_\varepsilon - 1)$,

(ii) $e = 3 + \sum_{\varrho=1}^{r} (b_\varrho - 2)$.

**Conclusion.** Statement (ii) gives together with Lemma 2 the following familiar formula (viz. [1], p. 349):

\[
\text{embdim } A_{n,q} = 3 + \sum_{\varrho=1}^{r} (b_\varrho - 2).
\]

4. Relations

Let $X_{n,q}$, $A_{n,q} = K_e/a_{n,q}$ from now on. To obtain a deformation of $X_{n,q}$, one must lay out a relation module of $a_{n,q}$. We set (under variable indices)

\[
g_{ij} = x_i x_j - p_{ij}, \quad 2 \leq i + 1 = j - 1 \leq e - 1,
\]

with $p_{ij}$ as in (7), and we must thus find all $(R_{ij})_{2 \leq i+1 = j-1 \leq e-1} \in e_1 K_e$ with

\[
\sum R_{ij} g_{ij} = 0.
\]
Theorem 6. The relation module of $a_{n,q}$ is generated by the following relations:

\begin{align*}
x_j g_k &= x_i g_{j,k} + \left(x_{i+1}^{a_{i+1}-2} \cdots x_k^{a_k-2} \right) x_k^{a_k-1} g_{i,j+1}, \quad 1 \leq i < j < k - 1 \leq e - 1, \\
x_j g_k &= x_k g_{j} + x_{i+1}^{a_{i+1}-1} \left(x_{i+2}^{a_{i+2}-2} \cdots x_j^{a_j-1} \right) g_{j,k-1}, \quad 2 \leq i + 1 < j < k \leq e,
\end{align*}

for which the brackets in the cases $j = k - 2$ resp. $i = j - 2$ go away.

Proof. Omitted. See original. \hfill \square

Conclusion. One can show that the now more construct minimal resolution

\[ e_2 K_e \to e_1 K_e \to K_e \to K_e / a_{n,q} \to 0 \]

is in mind of [4, Chapter III, §2.1], where $e_2$ is the number of relations in Theorem 6:

\[ e_2 = \frac{1}{3}(e - 1)(e - 2)(e - 3). \]

That $A_{n,q} = K_e / a_{n,q}$ is normal and 2-dimensional and therefore also Macaulay (which also follows from $A_{n,q} = K_2^{G_{n,q}}$), shows to the Syzygy theorem (viz. e.g. [4, Theorem III.2.6]) for the syzygy length of $A_{n,q}$:

\[ \text{syl}_K A_{n,q} = e - 2. \]

In the case $e = 4$ the map $e_2 K_e \to e_1 K_e$ is already injective, and therefore

\[ 0 \to 2 K_4 \to 3 K_4 \to K_4 / a_{n,q} \to 0 \]

is a Hilbert resolution of $K_4 / a_{n,q}$. In the homogeneous case

\[ a_{3,1} = (x_1^2, x_2^2, x_3, x_1 x_4 - x_2 x_4)k\langle x_1, x_2, x_3, x_4 \rangle \]

one can find this exact sequence of Hilbert ([5], p. 503).

It is supposed that for all Hilbert resolutions for any $e$ always the following holds:

\[ 0 \to e_{e-2} K_e \to e_{e-3} K_e \to \cdots \to e_2 K_e \to e_1 K_e \to K_e \to K_e / a_{n,q} \to 0, \]

where

\[ e_j = \frac{1}{j + 1}(e - 1)(e - 2) \cdots (e - (j + 1)), \quad 1 \leq j \leq e - 2. \]

References