ERRATA AND ADDENDA: *QUATERNION ALGEBRAS*

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This note gives some errata and addenda for the first edition, 2021 printing, of *Quaternion algebras* [1]. Thanks to Sarah Arpin, Eran Assaf, Angie Babei, Enzo Brechler, Nils Bruin, Louis Carlin, Adam Chapman, Nirvana Coppola, Rob de Jeu, Michel Duneau, Manoj Keshari, T.-Y. Lam, Stefano Marseglia, Kimball Martin, Aurel Page, Francesco Sica, Harry Smit, Kate Stange, and Jiangwei Xue.

Errata

Mathematical glitches and errors.

(1) (2.4.3): the second equality only holds for $n = 2$, so replace with "where

$$\mathrm{SU}(n) := \{A \in \mathrm{SL}_n(\mathbb{C}) : A^* = A^{-1}\}$$

and $A^* = \overline{A}^{\mathsf{t}}$ is the (complex) conjugate transpose of A."

(2) Proof of Lemma 3.4.2, line 5: "B" should be "K".

(3) Section 4.1, page 48, line 1, "$\det f$" should be "$\det f = 1$".

(4) 4.2.21: replace with "$\langle a_1, \dots, a_n\rangle := \langle a_1\rangle \boxplus \cdots \boxplus \langle a_n\rangle$" (swap sides).

(5) 4.5.8: "Then there is a" should be "Writing $V = B^0$, there is a"; replace $\mathrm{trd}(B)$ with $\mathrm{trd}(v)$"; and in (4.5.9) replace "B^0" by "V".

(6) Example 4.5.13: "$\det^0$" should be "$\det|_{B^0}$".

(7) Example 4.5.14: "det" should be "nrd".

(8) Proof of Proposition 4.5.17: "$x\overline{x}\,\overline{x}^{-1}$" should be "$x\overline{v}\,\overline{x}^{-1}$".

(9) Beginning of section 5.3: we have only defined quadratic forms when $\mathrm{char}\, F \neq 2$. Replace "we pause our assumption and allow F of arbitrary characteristic" with "the reader may continue to suppose that $\mathrm{char}\, F \neq 2$, but the constructions in this section work quite generally, so the reader may also wish to return to this section after reading Chapter 6 and allow $\mathrm{char}\, F = 2$."

(10) Proof of Lemma 5.3.14: the one given only works when $\mathrm{char}\, F \neq 2$. Add "We give a proof when $\mathrm{char}\, F \neq 2$; for another approach that works more generally, see Exercise 5.20" to the start of the proof, and remove the parenthetical clause "(Alternatively, this can be viewed as a graded tensor product; see Exercise 5.21)."

(11) Proof of Lemma 5.4.2: "$y \leftarrow y - 2x/T(y, y)$" should be "$y \leftarrow y - T(y, y)x/2$".

(12) Proof of Lemma 5.4.7: "$\gamma = \alpha\beta^{-1}$" should be "$\gamma = -\alpha\beta^{-1}$".

(13) 5.6.7: "δ" should be "ζ".

(14) (5.6.10): "$-\,\mathrm{nrd}(v)$" should be "$Q(v)$".

(15) Proof of Theorem 6.4.7: $Q \sim \langle 1\rangle$ should be $Q \simeq \langle 1\rangle$.

(16) Proof of Theorem 7.1.5: "$\beta \in B \otimes_F F^{\mathrm{sep}} \simeq \mathrm{GL}_2(F^{\mathrm{sep}})$" should be "$\beta \in (B \otimes_F F^{\mathrm{sep}})^\times \simeq \mathrm{GL}_2(F^{\mathrm{sep}})$".

(17) Proof of Theorem 7.3.5(b): Here is a better proof for the first part. "From (a), we have $V = \sum_i V_i$ the sum of simple B-modules, so $W = \sum_i (V_i \cap W)$. Since each V_i is simple, we

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have $V_i \cap W = \{0\}$ or $V_i \cap W = V_i$, and either way we conclude that W is a sum of simple modules. So by (a), W is semisimple."

(18) Corollary 7.7.11: should assume B is a division algebra. Replace the statement with "Let B be a central division F-algebra and let K be a maximal subfield. Then $\dim_F B = (\dim_F K)^2$." And the proof should read "Since B is a division algebra and K is maximal subfield, in fact K is a maximal commutative F-subalgebra, so $C_B(K) = K$ and thus by Proposition 7.7.8(b) we have $\dim_F B = (\dim_F K)^2$."

(19) 11.5.7: the vertices given are for the dodecahedron.

(20) Proposition 15.6.7: all occurrences of $\mathfrak{a}$ should be replaced by R. The whole point of taking trace duals is to have $\operatorname{trd}(\alpha\beta) \in R$ for $\alpha \in I$ and $\beta \in I^\sharp$!

(21) §16.1, "and the product of two (say) right $\mathcal{O}$-ideals need not be again a right $\mathcal{O}$-ideal! To address this, for lattices I, J": in any ring A, the product of two right A-ideals is again an A-ideal! (There is a problem with the product of two *locally principal* right $\mathcal{O}$-ideals from being again locally principal, but it is too soon to say that. We also have that the product of a right $\mathcal{O}$-ideal and a left $\mathcal{O}$-ideal need not be left or right $\mathcal{O}$-ideal.) Replace with "To study ideals of $\mathcal{O}$ we must distinguish between left or right ideals and take care with products. For lattices I, J".

(22) Proof of Main Theorem 16.6.1: for the application to Proposition 16.6.15(a), we need $I^3 = I^4$, which here reads $I^{n-1} = I^n$. This is obtained by taking $\alpha_1 = 1$, which can be justified as follows: We may suppose without loss of generality that $\alpha_1 = 1$: indeed, if $\mathfrak{p}$ is the maximal ideal of R and $k := R/\mathfrak{p}$ its residue field, then $I/\mathfrak{p}I \simeq k^n$ is a k-vector space with $1 \neq 0$, so we can extend to a basis and this lifts to a basis over R, by Nakayama's lemma. In the rest of the proof, replace n by $n-1$.

(23) Lemma 17.3.3: for (iii), also require "If further I, J are invertible with $\mathcal{O}_{\mathsf{R}}(I) = \mathcal{O}_{\mathsf{R}}(J)$".

(24) Lemma 17.7.26: add "I" to the statement and change the proof to read: "For such $I \subseteq \mathcal{O}$, we have $\mathsf{N}(I) = [\mathcal{O} : I]_{\mathbb{Z}} \leq C$, the index taken as abelian groups. But there are only finitely many subgroups of $\mathcal{O}$ of index $\leq C$, since $\mathcal{O}$ is finitely generated: they correspond to the possible kernels of surjective group homomorphisms $\mathcal{O} \to A$ where $\#A = n \leq C$."

(25) Proposition 19.4.1: add as the first sentence of the proof: "Multiplication is defined by 16.5.3."

(26) Proof of Theorem 20.3.3, before (20.3.5): "to show that I is left invertible" should be "to show that $I^{-1}I = \mathcal{O}_{\mathsf{R}}(I)$".

(27) After (26.6.6), add "When $(\mathcal{O} \mid \mathfrak{p}) = *$, we define $\lambda(\mathcal{O}, \mathfrak{p}) = 1$."

(28) Theorem 28.5.5: "$\operatorname{nrd}(\mathcal{O}^\times)$" should be "$\operatorname{nrd}(\widehat{\mathcal{O}}^\times)$".

(29) 31.1.19: the equality between reduced norm and index is not true in general. The statement (Proposition 31.4.4) holds for $\mathfrak{a} = \operatorname{nrd}(J)$. One can work with the index with the following additional clause: "Without loss of generality, by weak approximation we may suppose that $\mathcal{O}'_\mathfrak{p} = \mathcal{O}_\mathfrak{p}$ for all $\mathfrak{p}$ dividing the level $\mathfrak{M}$ of $\mathcal{O}$ and $\mathcal{O}'$."

(30) Proof of Lemma 42.2.13: the expression for $E[I\beta]$ holds when $\operatorname{nrd}(I\beta)$ is coprime to p; otherwise, this should be interpreted as a scheme-theoretic kernel.

(31) Proof of Proposition 42.2.16(b), "same right $\mathcal{O}'$-ideal class" should be "same left $\mathcal{O}'$-ideal class".

(32) Proof of Lemma 42.2.7: factoring through ϕ_I is not the definition of I! This statement follows from Proposition 42.2.16(b), so one could borrow from the future. Or see the addenda item below.

(33) (42.2.18): "$\operatorname{rk} E[I']$" should be "$\operatorname{rk} E_I[I']$".

(34) Proof of Theorem 42.3.2, "Tensoring with $\mathbb{Q}$... we may suppose $I_0 \subseteq B_0$": all occurrences of I_0 should be replaced by I.

(35) (42.3.5): "$(I' : I)$" should be "$(I : I')$".

(36) 43.5.7: I_6 is not holomorphic! So replace with "The functions I_4, I_{10} are holomorphic, but I_2, I_6 are meromorphic (poles as in Lemma 43.5.5)."
(37) 43.5.9: In the Albert classification, B is simple, so case (v) should not occur (and in case (iii), the quaternion algebra is a division algebra), so "five cases" should be "cases". See also the addenda below, which describes the split cases as well.

Exercises.

(1) Exercise 2.11: formatting on a) and b) is wrong, should match (a) and (b) in Exercise 2.9.
(2) Exercise 2.16: "βw" should be "$\mathrm{tr}(\lambda(\beta w))$".
(3) Exercise 3.6: "subfields" should be "quadratic subfields (over F)". (One does not need B to be a division quaternion algebra for the first statement.)
(4) Exercise 3.14: should be "$\mathrm{trd}(\alpha)$" not "$\mathrm{trd}(A)$".
(5) Exercise 3.18: replace "$V(B) =$" with "$V(B) :=$", and replace last two sentences "Let B be a ... over F" with "Let B be a *central* division ring over F. Show that $V(B)$ is a nonzero vector space if and only if B is a quaternion algebra over F."
(6) Exercise 4.5: V should be nondegenerate.
(7) Exercise 4.7: in (a), matrix should be transposed to get a left action; in (b), replace "$A[T]A^{\mathsf{t}}$" with "$A^{\mathsf{t}}[T]A$".
(8) Exercise 4.8(a): need $i' \neq 0$.
(9) Exercise 5.7: delete "$(-1, 26)_{\mathbb{Q}} = 1$, i.e.,", so the exercise is "Show $\left(\dfrac{-1, 26}{\mathbb{Q}}\right) \simeq \mathrm{M}_2(\mathbb{Q})$."
(10) Exercise 5.10(a): replace "$k \in \{i, j, ij\}$" with "$k \in B^0$". (Or keep this as is, then you can take $t = 0$ in the formulas after.)
(11) Exercise 5.12: replace "$^-\colon \mathrm{Clf}^0\, Q \to \mathrm{Clf}^0\, Q$" with "$^-\colon \mathrm{Clf}\, Q \to \mathrm{Clf}\, Q$".
(12) Exercise 5.15: move to end of chapter 12.
(13) Exercise 5.22: "R-algebra" should be "F-algebra".
(14) Exericse 5.23: "an linear" should be "a linear".
(15) Exercise 6.12: "$\zeta^2 = 1$" should be "$\zeta^2 = d$".
(16) Exercise 7.6: "simple F-algebra".
(17) Exercise 7.8: "$(K \mid b)$" should be "$(K, b \mid F)$".
(18) Exercise 7.10: "show" should be "show directly".
(19) Exercise 7.15(c): The summation should be over $g \in G$, "g^{-1}" should be "$(g^{-1})^{\mathrm{o}}$", and "Give B the structure of a B^{e}-algebra" should be "Give B the structure of a B^{e}-module".
(20) Exercise 7.20: add "(viz. Main Theorem 4.4.1)" at end.
(21) Exercise 7.23: "Exercise 7.18" should be "Exercise 7.17".
(22) Exercise 7.24: "let $f(T) \in K[T]$" should be "let $f(T) \in K[T]$ be monic".

Typos/copyediting.

(1) Section 2.1, line 5: replace with "$G^n := \{g^n : g \in G\} \leq G$ for the subgroup of nth powers".
(2) Section 2.1, line 7: space missing in "B equipped".
(3) Section 2.1, page 22, line 2: replace "notation. reserve" with "notation. We reserve".
(4) Section 2.1, page 22, line 4: replace "$\mathrm{End}_F(B) \sim \mathrm{M}_n(F)$" with "$\mathrm{End}_F(B) \simeq \mathrm{M}_n(F)$".
(5) 3.2.9, line 7: delete extraneous "$\overline{ij}$".
(6) Section 4.1, line 10: "element respect to" should be "element with respect to".
(7) Exercise 4.4: a) and b) should be (a) and (b).
(8) Exercise 4.8: "readers some" should be "readers with some".
(9) Exercise 4.16(a): in hint, "complement of V" should be "complement in V".
(10) (5.3.3): line after, "two-sided ideal generated the" should be "two-sided ideal generated by the".

(11) Definition 4.2.12: delete preceding sentence "From now on... associated to $\mathbb{Q}$".
(12) Remark 4.2.19: "as least as old" could be "at least as old".
(13) Exercise 5.2: second "(a)" should be "(b)".
(14) Section 6.1, line 6, "But always have scalar norms" should be "But we always have scalar norms".
(15) Section 7.7: Replace "We conclude this chapter with" with "In this section, we establish".
(16) Corollary 7.7.6: in the line before, replace "*isomorphism classes* of quaternion algebras" with "*isomorphism classes* of quaternion algebras (also proven in Exercise 6.4, in a different way)."
(17) 7.7.12: "Exercise 7.11" should be "Exercise 7.10".
(18) Section 7.8: "In this last section, we" should be "We now", and delete "Thr".
(19) Proof of Lemma 7.8.5: delete "For part (a)".
(20) Lemma 7.8.8: "(as in the proof of Lemma 7.8.5" needs right parenthesis.
(21) Before 8.2.7: "Laghbribi" should be "Laghribi".
(22) Remark 8.2.9: "[Lam2005, Example VI.1]" should be "[Lam2005, Example VI.1.11]".
(23) Before Theorem 13.3.11: add "We recall the notation 6.1.5."
(24) p. 258, "both of these products are compatible": delete "are".
(25) Remark 15.6.18, "instead of the codifferent instead a *different*": delete second "instead", add a comma.
(26) Before 16.2.5, replace "$B = FJ = F(rJ) \subseteq FIJ$" with "$B = F(rJ) \subseteq F(IJ) = B$"; and in the line before, replace "$r \in I$" with "$r \in R \cap I$".
(27) p. 266, line -2, "Finally, not every lattice": delete "Finally".
(28) p. 280, line -2, "Lemma 17.3.3(b)": should be "Lemma 17.3.3(ii)".
(29) 17.3.7: specify that $B = \mathrm{M}_n(F)$.
(30) 17.4.15, "is given": should be "are given".
(31) p. 312, middle paragraph: missing parenthesis at end.
(32) Before Remark 19.5.8: "Brant" should be "Brandt".
(33) Before Theorem 20.1.1: delete extra space before "**projective**".
(34) Proof of Theorem 20.3.3: The two occurrences of "α_i" indicating a set should be "$\{\alpha_i\}_i$".
(35) Remark 20.3.6, line 2: delete extra space before "**dual basis lemma**".
(36) 23.2.2: "have the nice local description" should be "have the following nice local description".
(37) Definition 24.3.2: space missing before "residually" in two places.
(38) Proof of Lemma 26.6.7: space missing in "that if($\mathcal{O}'$".
(39) After (41.5.6): delete indent in front of "where $\delta = 1, 0$".
(40) After (42.2.2): "$E[\alpha] = \ker \alpha$" should be "$E[\alpha] := \ker \alpha$".
(41) Proof of Lemma 42.2.22: at the end of the proof, ".." should be ".".
(42) (42.2.27): "$\mathrm{Hom}(E_{I'} E_I)$" should be "$\mathrm{Hom}(E_{I'}, E_I)$".
(43) Bibliography: the items [Hur1896] and [Hur1898] should be interchanged.

Addenda

(1) Example 28.5.20: Let F be a number field and let B be an indefinite quaternion algebra over F (so either F has a complex place or at least one real place of F is unramified in B). Suppose that $R = \mathbb{Z}_F$ has narrow class number 1, and let $\mathcal{O} \subseteq B$ be an Eichler R-order in B. Then $\#\,\mathrm{Cls}\,\mathcal{O} = 1$. Indeed, we apply Theorem 28.5.5: by Example 28.5.16, the order $\mathcal{O}$ is locally norm-maximal so $\mathrm{Cl}_{G(\mathcal{O})}\, R$ is a quotient of the narrow class group, which is trivial.
(2) Lemma 42.2.7 can be proven by appeal to the Isogeny Theorem, as follows.

The image of $\operatorname{Hom}(E_I, E)$ under precomposition by ϕ_I lands in $\operatorname{End}(E) = \mathcal{O}$. We check locally that the image is I. First, we may replace I by an ideal in the same left $\mathcal{O}$-ideal class to suppose that $\operatorname{nrd}(I)$ is coprime to p. Then $I_p = \mathcal{O}_p$. For the remaining primes, let $\ell \neq p$ be prime. As in the proof of Lemma 42.1.11, the Isogeny Theorem gives

$$\operatorname{Hom}(E_I, E) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} \operatorname{Hom}(T_\ell(E_I), T_\ell(E))$$

(recalling that over a sufficiently large finite subfield of F, the Galois action is scalar). Since I is locally principal, we have $I_\ell = \mathcal{O}_\ell \alpha_\ell$ for some $\alpha_\ell \in \mathcal{O}_\ell \simeq \mathrm{M}_2(\mathbb{Z}_\ell)$ with $T_\ell(E) = \mathbb{Z}_\ell^2$. Then $T_\ell(E_I) = \alpha_\ell^{-1} T_\ell(E)$ and so

$$\operatorname{Hom}(T_\ell(E_I), T_\ell(E)) = \mathcal{O}_\ell \alpha_\ell.$$

The pullback map

$$\operatorname{Hom}(T_\ell(E_I), T_\ell(E)) \to \mathcal{O}_\ell$$

is just the identity map, since we are already writing isogenies with respect to the fixed (standard) basis of $T_\ell(E)$; so its image is $\mathcal{O}_\ell \alpha_\ell = I_\ell$. Therefore the image lies in I by the local–global dictionary for lattices.

(3) Remark 8.2.9: Albert's book [Alb39] on algebras still reads well today. The proof of the key implication (iii) $\Rightarrow$ (i) in Proposition 8.2.3 is due to him [Alb72]. ("I discovered this theorem some time ago. There appears to be some continuing interest in it, and I am therefore publishing it now.") Albert [Alb32] used Proposition 8.2.8 to show that over $F = \mathbb{R}(x, y)$, the tensor product of

$$B_1 = \left(\frac{x, -1}{F}\right) \quad \text{and} \quad B_2 = \left(\frac{-x, y}{F}\right)$$

is a division algebra, by verifying that the Albert form $Q(B_1, B_2)$ is anisotropic over F. See Lam [Lam2005, Albert's Theorem 4.8, Example VI.1.11] for more details.

For the fields of interest in this book (local fields and global fields), a biquaternion algebra will never be a division algebra—the proof of this fact rests on classification results for quaternion algebras over these fields, which we will take up in earnest in Part II.

(4) Exercise 42.5: In the proof of Proposition 42.2.16, we considered $II' = \mathcal{O}\alpha$ and the isogeny $\phi_{I'} \colon E_I \to E_I/E_I[I']$, which moves away from the setup with the fixed supersingular elliptic curve E. We may proceed differently as follows.
   (a) Let $m := \operatorname{nrd}(I)$. From $I\overline{I} = \mathcal{O}m$ show that $\phi_{\overline{I}} = \phi_I^\vee$ (dual isogeny). Conclude that $\deg \phi_I = \deg \phi_{\overline{I}}$.
   (b) Prove $\deg \phi_{I'} \mid \operatorname{nrd}(I')$ by working with $\phi_{\overline{I'}} \colon E \to E_{\overline{I'}}$.

(5) 43.5.9: Replace with the following.

Let A be a principally polarized complex abelian surface. Let $\operatorname{End}(A)$ be the ring of endomorphisms of A, and let $B = \operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. If $A \sim E_1 \times E_2$ is isogenous to the product of two elliptic curves, then either $E_1 \not\sim E_2$ are not isogenous and $B \simeq \operatorname{End}(E_1) \times \operatorname{End}(E_2)$ or $E_1 \sim E_2 \sim E$ and $B \simeq \operatorname{End}(E^2) \simeq \mathrm{M}_2(\operatorname{End}(E))$. As the endomorphism algebra of an elliptic curve is either $\mathbb{Q}$ or an imaginary quadratic field K, this gives four possibilities: $B \simeq \mathbb{Q} \times \mathbb{Q}, \mathbb{Q} \times K, \mathrm{M}_2(\mathbb{Q}), \mathrm{M}_2(K)$. Otherwise, B is simple, and by the classification theorem of Albert (Theorem 8.5.4), the $\mathbb{Q}$-algebra B is exactly one of the following:
   (a) $B = \mathbb{Q}$, and we say A is **typical**;
   (b) $B = F$ a real quadratic field, and we say A has **real multiplication (RM)** by F;
   (c) B is an indefinite division quaternion algebra over $\mathbb{Q}$, and we say A has **quaternionic multiplication (QM)** by B; or
   (d) $B = K$ is a quartic CM field K, and we say A has **complex multiplication (CM)** by K.

One may also view the products $B \simeq \mathbb{Q} \times \mathbb{Q}$ and $B \simeq \mathrm{M}_2(\mathbb{Q})$ as special cases of (ii) and (iii), respectively.

References

[1] John Voight, *Quaternion algebras*, Grad. Texts in Math., vol. 288, Springer, Cham, 2021.