ERRATA AND ADDENDA: *QUATERNION ALGEBRAS*

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Errata

Mathematical glitches and errors.

(1) (2.4.3): the second equality only holds for $n = 2$, so replace with "where

$$\mathrm{SU}(n) := \{A \in \mathrm{SL}_n(\mathbb{C}) : A^* = A^{-1}\}$$

and $A^* = \overline{A}^{\mathsf{t}}$ is the (complex) conjugate transpose of A."

(2) Proof of Lemma 3.4.2, line 5: "B" should be "K".

(3) Section 4.1, page 48, line 1, "$\det f$" should be "$\det f = 1$".

(4) 4.2.20: should be $Q'(x')$ and $T'(x', y')$.

(5) 4.2.21: replace with "$\langle a_1, \dots, a_n\rangle := \langle a_1\rangle \boxplus \cdots \boxplus \langle a_n\rangle$" (swap sides).

(6) 4.5.8: "Then there is a" should be "Writing $V = B^0$, there is a"; replace $\mathrm{trd}(B)$ with $\mathrm{trd}(v)$"; and in (4.5.9) replace "B^0" by "V".

(7) Example 4.5.13: "$\det^0$" should be "$\det|_{B^0}$".

(8) Example 4.5.14: "det" should be "nrd".

(9) Proof of Proposition 4.5.17: "$x\overline{x}\,\overline{x}^{-1}$" should be "$x\overline{v}\,\overline{x}^{-1}$".

(10) Beginning of section 5.3: we have only defined quadratic forms when $\mathrm{char}\, F \neq 2$. Replace "we pause our assumption and allow F of arbitrary characteristic" with "the reader may continue to suppose that $\mathrm{char}\, F \neq 2$, but the constructions in this section work quite generally, so the reader may also wish to return to this section after reading Chapter 6 and allow $\mathrm{char}\, F = 2$."

(11) Proof of Lemma 5.3.14: the one given only works when $\mathrm{char}\, F \neq 2$. Add "We give a proof when $\mathrm{char}\, F \neq 2$; for another approach that works more generally, see Exercise 5.20" to the start of the proof, and remove the parenthetical clause "(Alternatively, this can be viewed as a graded tensor product; see Exercise 5.21)."

(12) Proof of Lemma 5.4.2: "$y \leftarrow y - 2x/T(y,y)$" should be "$y \leftarrow y - T(y,y)x/2$".

(13) Proof of Lemma 5.4.7: "$\gamma = \alpha\beta^{-1}$" should be "$\gamma = -\alpha\beta^{-1}$".

(14) 5.6.7: "δ" should be "ζ".

(15) (5.6.10): "$-\mathrm{nrd}(v)$" should be "$Q(v)$".

(16) Proof of Theorem 6.4.7: $Q \sim \langle 1\rangle$ should be $Q \simeq \langle 1\rangle$.

(17) Proof of Theorem 7.1.5: "$\beta \in B \otimes_F F^{\mathrm{sep}} \simeq \mathrm{GL}_2(F^{\mathrm{sep}})$" should be "$\beta \in (B \otimes_F F^{\mathrm{sep}})^\times \simeq \mathrm{GL}_2(F^{\mathrm{sep}})$".

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(18) Proof of Theorem 7.3.5(b): Here is a better proof for the first part. "From (a), we have $V = \sum_i V_i$ the sum of simple B-modules, so $W = \sum_i (V_i \cap W)$. Since each V_i is simple, we have $V_i \cap W = \{0\}$ or $V_i \cap W = V_i$, and either way we conclude that W is a sum of simple modules. So by (a), W is semisimple."
(19) Corollary 7.7.11: should assume B is a division algebra. Replace the statement with "Let B be a central division F-algebra and let K be a maximal subfield. Then $\dim_F B = (\dim_F K)^2$." And the proof should read "Since B is a division algebra and K is maximal subfield, in fact K is a maximal commutative F-subalgebra, so $C_B(K) = K$ and thus by Proposition 7.7.8(b) we have $\dim_F B = (\dim_F K)^2$."
(20) Below (11.2.8): replace with "$s = -\omega^2 = (1 + i + j + k)/2$, and $t = (1 + i - j + k)/2$".
(21) Proof of Proposition 11.3.4: replace "$\alpha = \mu\beta + \rho$" with "$\alpha = \beta\mu + \rho$".
(22) Remark 11.4.11: FGS should be [FGS2016].
(23) 11.5.7: the vertices given are for the dodecahedron.
(24) Proposition 15.6.7: all occurrences of $\mathfrak{a}$ should be replaced by R. The whole point of taking trace duals is to have $\operatorname{trd}(\alpha\beta) \in R$ for $\alpha \in I$ and $\beta \in I^\sharp$!
(25) §16.1, "and the product of two (say) right $\mathcal{O}$-ideals need not be again a right $\mathcal{O}$-ideal! To address this, for lattices I, J": in any ring A, the product of two right A-ideals is again an A-ideal! (There is a problem with the product of two *locally principal* right $\mathcal{O}$-ideals from being again locally principal, but it is too soon to say that. We also have that the product of a right $\mathcal{O}$-ideal and a left $\mathcal{O}$-ideal need not be left or right $\mathcal{O}$-ideal.) Replace with "To study ideals of $\mathcal{O}$ we must distinguish between left or right ideals and take care with products. For lattices I, J".
(26) Example 16.5.12: the last equality in (16.5.14) and the final equality is wrong, since $1/p$ is not in any order! Should be $\mathbb{Z} + \dfrac{1}{p}\mathcal{O}^0$.
(27) Remark 16.5.19: confusion with d versus d_K, should read "as abelian groups, we have

$$\mathfrak{f} = f\mathbb{Z} + \sqrt{d}\mathbb{Z} = f \cdot S(d_K),$$

so $\mathfrak{f}$ is principal and hence certainly invertible as an ideal of $S(d_K)$—but not as an ideal of the smaller order $S(d)$."
(28) Proof of Main Theorem 16.6.1: for the application to Proposition 16.6.15(a), we need $I^3 = I^4$, which here reads $I^{n-1} = I^n$. This is obtained by taking $\alpha_1 = 1$, which can be justified as follows: We may suppose without loss of generality that $\alpha_1 = 1$: indeed, if $\mathfrak{p}$ is the maximal ideal of R and $k := R/\mathfrak{p}$ its residue field, then $I/\mathfrak{p}I \simeq k^n$ is a k-vector space with $1 \neq 0$, so we can extend to a basis and this lifts to a basis over R, by Nakayama's lemma. In the rest of the proof, replace n by $n - 1$.
(29) Lemma 17.3.3: for (iii), also require "If further I, J are invertible with $\mathcal{O}_{\mathsf{R}}(I) = \mathcal{O}_{\mathsf{R}}(J)$".
(30) Lemma 17.7.26: add "I" to the statement and change the proof to read: "For such $I \subseteq \mathcal{O}$, we have $\mathsf{N}(I) = [\mathcal{O} : I]_{\mathbb{Z}} \leq C$, the index taken as abelian groups. But there are only finitely many subgroups of $\mathcal{O}$ of index $\leq C$, since $\mathcal{O}$ is finitely generated: they correspond to the possible kernels of surjective group homomorphisms $\mathcal{O} \to A$ where $\#A = n \leq C$."
(31) Proposition 19.4.1: add as the first sentence of the proof: "Multiplication is defined by 16.5.3."
(32) Proof of Theorem 20.3.3, before (20.3.5): "to show that I is left invertible" should be "to show that $I^{-1}I = \mathcal{O}_{\mathsf{R}}(I)$".
(33) Proposition 24.5.14(a): 'ramified' and 'split' are reversed.
(34) After (26.6.6), add "When $(\mathcal{O} \mid \mathfrak{p}) = *$, we define $\lambda(\mathcal{O}, \mathfrak{p}) = 1$."
(35) Theorem 28.5.5: "$\operatorname{nrd}(\mathcal{O}^\times)$" should be "$\operatorname{nrd}(\widehat{\mathcal{O}}^\times)$".

(36) Proposition 30.7.4: the proof of Theorem 30.4.7 does say how to handle the normalizer group, and in fact it can be quite complicated! This should be deleted, and to keep numbering consistent, the first equation in the example that follows should be numbered.
(37) 31.1.19: the equality between reduced norm and index is not true in general. The statement (Proposition 31.4.4) holds for $\mathfrak{a} = \operatorname{nrd}(J)$. One can work with the index with the following additional clause: "Without loss of generality, by weak approximation we may suppose that $\mathcal{O}'_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}$ for all $\mathfrak{p}$ dividing the level $\mathfrak{M}$ of $\mathcal{O}$ and $\mathcal{O}'$."
(38) 33.2.4: Replace "A **geodesic** is a continuous map $(-\infty, \infty) \to X$" with "A **geodesic** is the image of a continuous map $(-\infty, \infty) \to X$ such that the restriction to sufficiently small compact intervals defines a geodesic segment".
(39) Proof of Lemma 33.4.11: replace with "The proof is direct; it is requested in Exercise 33.5."
(40) Lemma 36.2.8: the proof is not 'identical', since we would need $c \in \mathbb{R}^\times$. But here is how to reduce to that case, replacing the proof: "Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$. We claim we may reduce to the case $c = 1$. Indeed, if $a = 0$, multiply on the left by an element of N to get $a \neq 0$; but then

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & (1-b)/a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -(1+c)/a & c \\ 1 & -a \end{pmatrix}.$$

Now repeat the first matrix calculation in Lemma 33.4.4, with $c = 1$."
(41) 36.5.1: angles sum to π, not 2π.
(42) 36.5.19: the series expansion for $\mathcal{L}(\theta)$ should have $(2\theta)^{2n}$ (not $2n+1$ in the exponent), and this equation should be numbered.
(43) 37.2.5: hyperbolic metric is missing a factor 2.
(44) Proof of Lemma 42.2.13: the expression for $E[I\beta]$ holds when $\operatorname{nrd}(I\beta)$ is coprime to p; otherwise, this should be interpreted as a scheme-theoretic kernel.
(45) Proof of Proposition 42.2.16(b), "same right $\mathcal{O}'$-ideal class" should be "same left $\mathcal{O}'$-ideal class".
(46) Proof of Lemma 42.2.7: factoring through ϕ_I is not the definition of I! This statement follows from Proposition 42.2.16(b), so one could borrow from the future. Or see the addenda item below.
(47) (42.2.18): "$\operatorname{rk} E[I']$" should be "$\operatorname{rk} E_I[I']$".
(48) Proof of Theorem 42.3.2, "Tensoring with $\mathbb{Q}$... we may suppose $I_0 \subseteq B_0$": all occurrences of I_0 should be replaced by I.
(49) (42.3.5): "$(I' : I)$" should be "$(I : I')$".
(50) 43.5.7: I_6 is not holomorphic! So replace with "The functions I_4, I_{10} are holomorphic, but I_2, I_6 are meromorphic (poles as in Lemma 43.5.5)."
(51) 43.5.9: In the Albert classification, B is simple, so case (v) should not occur (and in case (iii), the quaternion algebra is a division algebra), so "five cases" should be "cases". See also the addenda below, which describes the split cases as well.

Exercises.

(1) Exercise 2.11: formatting on a) and b) is wrong, should match (a) and (b) in Exercise 2.9.
(2) Exercise 2.16: "βw" should be "$\operatorname{tr}(\lambda(\beta w))$".
(3) Exercise 3.6: "subfields" should be "quadratic subfields (over F)". (One does not need B to be a division quaternion algebra for the first statement.)
(4) Exercise 3.14: should be "$\operatorname{trd}(\alpha)$" not "$\operatorname{trd}(A)$".
(5) Exercise 3.18: replace "$V(B) =$" with "$V(B) :=$", and replace last two sentences "Let B be a ... over F" with "Let B be a *central* division ring over F. Show that $V(B)$ is a nonzero vector space if and only if B is a quaternion algebra over F."

(6) Exercise 4.5: V should be nondegenerate.
(7) Exercise 4.7: in (a), matrix should be transposed to get a left action; in (b), replace "$A[T]A^{\mathsf{t}}$" with "$A^{\mathsf{t}}[T]A$".
(8) Exercise 4.8(a): need $i' \neq 0$.
(9) Exercise 5.7: delete "$(-1,26)_{\mathbb{Q}} = 1$, i.e.,", so the exercise is "Show $\left(\dfrac{-1,26}{\mathbb{Q}}\right) \simeq \mathrm{M}_2(\mathbb{Q})$."
(10) Exercise 5.10(a): replace "$k \in \{i,j,ij\}$" with "$k \in B^0$". (Or keep this as is, then you can take $t = 0$ in the formulas after.)
(11) Exercise 5.12: replace "$\overline{}\colon \operatorname{Clf}^0 Q \to \operatorname{Clf}^0 Q$" with "$\overline{}\colon \operatorname{Clf} Q \to \operatorname{Clf} Q$".
(12) Exercise 5.15: move to end of chapter 12.
(13) Exercise 5.22: "R-algebra" should be "F-algebra".
(14) Exericse 5.23: "an linear" should be "a linear".
(15) Exercise 6.12: "$\zeta^2 = 1$" should be "$\zeta^2 = d$".
(16) Exercise 7.6: "simple F-algebra".
(17) Exercise 7.8: "$(K \mid b)$" should be "$(K, b \mid F)$".
(18) Exercise 7.10: "show" should be "show directly".
(19) Exercise 7.15(c): The summation should be over $g \in G$, "g^{-1}" should be "$(g^{-1})^{\mathrm{o}}$", and "Give B the structure of a B^{e}-algebra" should be "Give B the structure of a B^{e}-module".
(20) Exercise 7.20: add "(viz. Main Theorem 4.4.1)" at end.
(21) Exercise 7.23: "Exercise 7.18" should be "Exercise 7.17".
(22) Exercise 7.24: "let $f(T) \in K[T]$" should be "let $f(T) \in K[T]$ be monic".
(23) Exercise 10.8: "$R = S[\alpha]$ a" should be "$R = S[\alpha]$ is a".
(24) Between Exercises 11.3 and 11.4: the one starting "Check that the map" should be a separate exercise; so the numbering of all of the remaining ones should increase by one.
(25) Exercise 11.8 (appears as 11.7): should be "such that $\|x - \lambda\|^2 \leq 1/2$".
(26) Exercise 11.9 (appears as 11.8): delete "definite", and replace "$\mathcal{O} \subseteq B$" with "$\mathcal{O} \subset B$".
(27) Exercise 11.10(c) (appears as 11.9(c)): replace exercise with: "More generally, if F is a field of characteristic 2 show that there is an exact sequence

$$1 \to F^2 \to \operatorname{Aut}_F(\mathcal{O} \otimes_{\mathbb{Z}} F) \to K^\times \rtimes \operatorname{Aut}_F(K) \to 1$$

where $K := F[\omega] \simeq F[x]/(x^2 + x + 1)$, and F^2 is considered as an additive group. *[Hint: let $J = \operatorname{rad}(\mathcal{O} \otimes_{\mathbb{Z}} F)$ be the Jacobson radical of the algebra, and show that the sequence is induced by an F-linear automorphisms of $K := F[\omega]$ and the automorphisms $\omega \mapsto \omega + \epsilon$ with $\epsilon \in J$.]*"
(28) Exercise 11.13 (appears as 11.12): replace "is conjugate in $\mathrm{O}(2)$ to" to "is".
(29) Exercise 11.16(b) (appears as 11.15(b)): replace "$x^2 + y^2 + z^2 = p$ with $x, y, z \in \mathbb{Z}$" with "$t^2 + x^2 + y^2 + z^2 = p$ with $t, x, y, z \in \mathbb{Z}$".
(30) Exercise 33.7: in the proof of Theorem 33.5.5, refer to Exercise 33.7(d), and replace the exercise as follows:

"In this exercise, we consider the action of $\mathrm{PSL}_2(\mathbb{R})$ on points and geodesics in $\mathbf{H}^2$.

(a) Show that $\mathrm{PSL}_2(\mathbb{R})$ acts transitively on the set of geodesics in $\mathbf{H}^2$.
(b) Show that $\mathrm{PSL}_2(\mathbb{R})$ acts transitively on the set of geodesics in $\mathbf{H}^2$ of a fixed length. *[Hint: using (a), reduce to the case where all four endpoints lie on the imaginary axis. Use elements of A in* (33.4.1) *to map one endpoint each to i; then use an element of K.]*
(c) Show that every orientation-preserving isometry of $\mathbf{H}^2$ that maps a geodesic to itself and fixes two points on this geodesic is the identity.
(d) Conclude that for every isometry ϕ of $\mathbf{H}^2$ and every geodesic in $\mathbf{H}^2$, there exists $g \in \mathrm{PSL}_2(\mathbb{R})$ such that $g\phi$ fixes the geodesic pointwise."

(31) Exercise 36.12: add new part (a), "Prove that $|B_{2k}| = (-1)^{k+1} B_{2k}$ for $k \geq 1$.

Typos/copyediting.

(1) Section 2.1, line 5: replace with "$G^n := \{g^n : g \in G\} \leq G$ for the subgroup of nth powers".
(2) Section 2.1, line 7: space missing in "B equipped".
(3) Section 2.1, page 22, line 2: replace "notation. reserve" with "notation. We reserve".
(4) Section 2.1, page 22, line 4: replace "$\mathrm{End}_F(B) \sim \mathrm{M}_n(F)$" with "$\mathrm{End}_F(B) \simeq \mathrm{M}_n(F)$".
(5) 3.2.9, line 7: delete extraneous "$\overline{ij}$".
(6) Section 4.1, line 10: "element respect to" should be "element with respect to".
(7) Exercise 4.4: a) and b) should be (a) and (b).
(8) Exercise 4.8: "readers some" should be "readers with some".
(9) Exercise 4.16(a): in hint, "complement of V" should be "complement in V".
(10) (5.3.3): line after, "two-sided ideal generated the" should be "two-sided ideal generated by the".
(11) Definition 4.2.12: delete preceding sentence "From now on... associated to $\mathbb{Q}$".
(12) Remark 4.2.19: "as least as old" could be "at least as old".
(13) Exercise 5.2: second "(a)" should be "(b)".
(14) Section 6.1, line 6, "But always have scalar norms" should be "But we always have scalar norms".
(15) Before Lemma 6.3.7, "Next, even though not every quadratic form" should be "Next, not every quadratic form".
(16) Section 7.7: Replace "We conclude this chapter with" with "In this section, we establish".
(17) Corollary 7.7.6: in the line before, replace "*isomorphism classes* of quaternion algebras" with "*isomorphism classes* of quaternion algebras (also proven in Exercise 6.4, in a different way)."
(18) 7.7.12: "Exercise 7.11" should be "Exercise 7.10".
(19) Section 7.8: "In this last section, we" should be "We now", and delete "Thr".
(20) Proof of Lemma 7.8.5: delete "For part (a)".
(21) Lemma 7.8.8: "(as in the proof of Lemma 7.8.5" needs right parenthesis.
(22) Before 8.2.7: "Laghbribi" should be "Laghribi".
(23) Remark 8.2.9: "[Lam2005, Example VI.1]" should be "[Lam2005, Example VI.1.11]".
(24) Proof of Lemma 11.1.2: Replace "Then $\mathrm{trd}(\alpha) = 2t \in \mathbb{Z}$, so by Corollary **??** we have $t \in \frac{1}{2}\mathbb{Z}$" by "Then $\mathrm{trd}(\alpha) = 2t \in \mathbb{Z}$ by Corollary 11.1.3, so $t \in \frac{1}{2}\mathbb{Z}$".
(25) Below Figure 11.2.7: replace "four inscribed tetrahedra" by "four inscribed regular tetrahedra".
(26) 11.3.1: replace "left (or right)" with "(left or) right".
(27) Before Proposition 11.3.4: replace "for division on the left" with "on the left".
(28) Before Corollary 11.3.6: replace "exists a greatest common divisor" with "exists a right greatest common divisor".
(29) Proof of Proposition 11.3.4: replace "left Euclidean" with "right Euclidean".
(30) Corollary 11.3.6: replace "there exists" with "there exist".
(31) (11.4.9): "f or" should be "for".
(32) Below (11.4.10): should be "$\gamma_1, \dots, \gamma_{r-1} \in \mathcal{O}^\times$".
(33) Before Theorem 13.3.11: add "We recall the notation 6.1.5."
(34) p. 258, "both of these products are compatible": delete "are".
(35) References to Exercise 13.7 (23.2.5, Exercise 23.8, 42.4.6) should be to Exercise 13.8.
(36) Remark 15.6.18, "instead of the codifferent instead a *different*": delete second "instead", add a comma.

(37) Before 16.2.5, replace "$B = FJ = F(rJ) \subseteq FIJ$" with "$B = F(rJ) \subseteq F(IJ) = B$"; and in the line before, replace "$r \in I$" with "$r \in R \cap I$".
(38) p. 266, line -2, "Finally, not every lattice": delete "Finally".
(39) p. 280, line -2, "Lemma 17.3.3(b)": should be "Lemma 17.3.3(ii)".
(40) 17.3.7: specify that $B = \mathrm{M}_n(F)$.
(41) 17.4.15, "is given": should be "are given".
(42) p. 312, middle paragraph: missing parenthesis at end.
(43) Before Remark 19.5.8: "Brant" should be "Brandt".
(44) Before Theorem 20.1.1: delete extra space before "**projective**".
(45) Proof of Theorem 20.3.3: The two occurrences of "α_i" indicating a set should be "$\{\alpha_i\}_i$".
(46) Remark 20.3.6, line 2: delete extra space before "**dual basis lemma**".
(47) 22.3.1 "nondegenerateternary" should be "nondegenerate ternary".
(48) 23.2.2: "have the nice local description" should be "have the following nice local description".
(49) Before (24.1.2), "proper implications": just "implications".
(50) Before (24.1.3), "$\mathcal{O}^\natural = \mathcal{O}_\mathsf{L}(\operatorname{rad}\mathcal{O})$": should be :=.
(51) Definition 24.3.2: space missing before "residually" in two places.
(52) Proof of Proposition 24.5.14: Delete duplicate reference to 24.5.12.
(53) Proof of Lemma 26.6.7: space missing in "that if($\mathcal{O}'$".
(54) Proof of Corollary 28.3.6: in the appeal to Lemma 28.7.2, add "borrowing (in a self-contained way) from the future".
(55) Proof of Proposition 36.6.2, "and $y' = y/\|z\|^2 > y$, and we repeat": replace with "and $y' = y/\|z\|^2 > y$, and so $\|\gamma z\|^2 \geq 1$".
(56) 37.2.9: in the subscript of $\bigcap$, replace $\Gamma-$ with $\Gamma\smallsetminus$.
(57) After (41.5.6): delete indent in front of "where $\delta = 1, 0$".
(58) After (42.2.2): "$E[\alpha] = \ker\alpha$" should be "$E[\alpha] := \ker\alpha$".
(59) Proof of Lemma 42.2.22: at the end of the proof, ".." should be ".".
(60) (42.2.27): "$\operatorname{Hom}(E_{I'}E_I)$" should be "$\operatorname{Hom}(E_{I'}, E_I)$".
(61) Bibliography: the items [Hur1896] and [Hur1898] should be interchanged.

Addenda

(1) Remark 8.2.9: Albert's book [Alb39] on algebras still reads well today. The proof of the key implication (iii) $\Rightarrow$ (i) in Proposition 8.2.3 is due to him [Alb72]. ("I discovered this theorem some time ago. There appears to be some continuing interest in it, and I am therefore publishing it now.") Albert [Alb32] used Proposition 8.2.8 to show that over $F = \mathbb{R}(x, y)$, the tensor product of

$$B_1 = \left(\frac{x, -1}{F}\right) \quad \text{and} \quad B_2 = \left(\frac{-x, y}{F}\right)$$

is a division algebra, by verifying that the Albert form $Q(B_1, B_2)$ is anisotropic over F. See Lam [Lam2005, Albert's Theorem 4.8, Example VI.1.11] for more details.

For the fields of interest in this book (local fields and global fields), a biquaternion algebra will never be a division algebra—the proof of this fact rests on classification results for quaternion algebras over these fields, which we will take up in earnest in Part II.

(2) Proof of Lemma 11.4.1: replace second sentence with: "Then $\mathcal{O}/p\mathcal{O} \simeq (-1, -1 \mid \mathbb{F}_p) \simeq \mathrm{M}_2(\mathbb{F}_p)$ by Wedderburn's little theorem. There exists a right ideal $I \bmod p \subset \mathcal{O}/p\mathcal{O}$ with $\dim_{\mathbb{F}_p}(I \bmod p) = 2$, for example $I \bmod p = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$."

(3) Example 28.5.20: Let F be a number field and let B be an indefinite quaternion algebra over F (so either F has a complex place or at least one real place of F is unramified in B). Suppose that $R = \mathbb{Z}_F$ has narrow class number 1, and let $\mathcal{O} \subseteq B$ be an Eichler R-order in B. Then $\#\operatorname{Cls}\mathcal{O} = 1$. Indeed, we apply Theorem 28.5.5: by Example 28.5.16, the order $\mathcal{O}$ is locally norm-maximal so $\operatorname{Cl}_{G(\mathcal{O})} R$ is a quotient of the narrow class group, which is trivial.

(4) Lemma 42.2.7 can be proven by appeal to the Isogeny Theorem, as follows.

The image of $\operatorname{Hom}(E_I, E)$ under precomposition by ϕ_I lands in $\operatorname{End}(E) = \mathcal{O}$. We check locally that the image is I. First, we may replace I by an ideal in the same left $\mathcal{O}$-ideal class to suppose that $\operatorname{nrd}(I)$ is coprime to p. Then $I_p = \mathcal{O}_p$. For the remaining primes, let $\ell \neq p$ be prime. As in the proof of Lemma 42.1.11, the Isogeny Theorem gives

$$\operatorname{Hom}(E_I, E) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} \operatorname{Hom}(T_\ell(E_I), T_\ell(E))$$

(recalling that over a sufficiently large finite subfield of F, the Galois action is scalar). Since I is locally principal, we have $I_\ell = \mathcal{O}_\ell \alpha_\ell$ for some $\alpha_\ell \in \mathcal{O}_\ell \simeq \mathrm{M}_2(\mathbb{Z}_\ell)$ with $T_\ell(E) = \mathbb{Z}_\ell^2$. Then $T_\ell(E_I) = \alpha_\ell^{-1} T_\ell(E)$ and so

$$\operatorname{Hom}(T_\ell(E_I), T_\ell(E)) = \mathcal{O}_\ell \alpha_\ell.$$

The pullback map

$$\operatorname{Hom}(T_\ell(E_I), T_\ell(E)) \to \mathcal{O}_\ell$$

is just the identity map, since we are already writing isogenies with respect to the fixed (standard) basis of $T_\ell(E)$; so its image is $\mathcal{O}_\ell \alpha_\ell = I_\ell$. Therefore the image lies in I by the local–global dictionary for lattices.

(5) Remark 33.2.8: at the end of the first paragraph, add "For an approach geared towards the context of hyperbolic geometry, see Ratcliffe [R."

(6) Remark 37.2.11: clarify first sentence "In the identification $\mathbf{H}^2 \to \mathbf{D}^2$, the preimage of an isometric circle in $\mathbf{D}^2$ is the corresponding perpendicular bisector, since this identification preserves hyperbolic distance."

(7) 37.3.3: replace first paragraph with: "We recall Definition 33.6.5 (sides and vertices) for hyperbolic polygons. For a Dirichlet domain ⌑, a side is a geodesic segment of positive length of the form $⌑ \cap \gamma ⌑$ with $\gamma \in \Gamma \smallsetminus \{1\}$; and a vertex is the point of intersection between two sides, equivalently, a vertex is a single point of the form $⌑ \cap \gamma ⌑ \cap \gamma' ⌑$ with $\gamma, \gamma' \in \Gamma$."

(8) Exercise 42.5: In the proof of Proposition 42.2.16, we considered $II' = \mathcal{O}\alpha$ and the isogeny $\phi_{I'} \colon E_I \to E_I/E_I[I']$, which moves away from the setup with the fixed supersingular elliptic curve E. We may proceed differently as follows.
   (a) Let $m := \operatorname{nrd}(I)$. From $I\overline{I} = \mathcal{O}m$ show that $\phi_{\overline{I}} = \phi_I^\vee$ (dual isogeny). Conclude that $\deg \phi_I = \deg \phi_{\overline{I}}$.
   (b) Prove $\deg \phi_{I'} \mid \operatorname{nrd}(I')$ by working with $\phi_{\overline{I'}} \colon E \to E_{\overline{I'}}$.

(9) 43.5.9: Replace with the following.

Let A be a principally polarized complex abelian surface. Let $\operatorname{End}(A)$ be the ring of endomorphisms of A, and let $B = \operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. If $A \sim E_1 \times E_2$ is isogenous to the product of two elliptic curves, then either $E_1 \not\sim E_2$ are not isogenous and $B \simeq \operatorname{End}(E_1) \times \operatorname{End}(E_2)$ or $E_1 \sim E_2 \sim E$ and $B \simeq \operatorname{End}(E^2) \simeq \mathrm{M}_2(\operatorname{End}(E))$. As the endomorphism algebra of an elliptic curve is either $\mathbb{Q}$ or an imaginary quadratic field K, this gives four possibilities: $B \simeq \mathbb{Q} \times \mathbb{Q}, \mathbb{Q} \times K, \mathrm{M}_2(\mathbb{Q}), \mathrm{M}_2(K)$. Otherwise, B is simple, and by the classification theorem of Albert (Theorem 8.5.4), the $\mathbb{Q}$-algebra B is exactly one of the following:
   (a) $B = \mathbb{Q}$, and we say A is **typical**;
   (b) $B = F$ a real quadratic field, and we say A has **real multiplication (RM)** by F;

(c) B is an indefinite division quaternion algebra over $\mathbb{Q}$, and we say A has **quaternionic multiplication (QM)** by B; or
(d) $B = K$ is a quartic CM field K, and we say A has **complex multiplication (CM)** by K.

One may also view the products $B \simeq \mathbb{Q} \times \mathbb{Q}$ and $B \simeq \mathrm{M}_2(\mathbb{Q})$ as special cases of (ii) and (iii), respectively.

References

[1] John Voight, *Quaternion algebras*, Grad. Texts in Math., vol. 288, Springer, Cham, 2021.