f-Continuous Operators on $B(L^2M)$ and the Families Index

Kameron McCombs

Dartmouth College

August 8, 2023

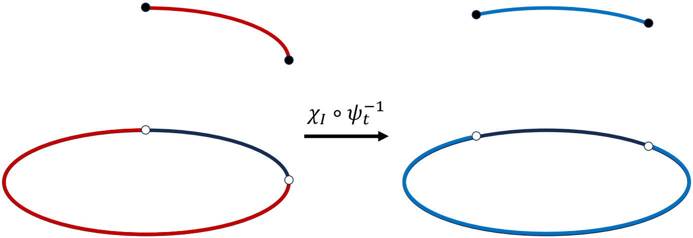
The Problem

If $M \to Z \xrightarrow{\pi} Y$ is a smooth fiber bundle with a family of operators $T = \{T_y\}$ on each fiber $L^2 Z_y$, what does it mean for this family to be norm continuous?

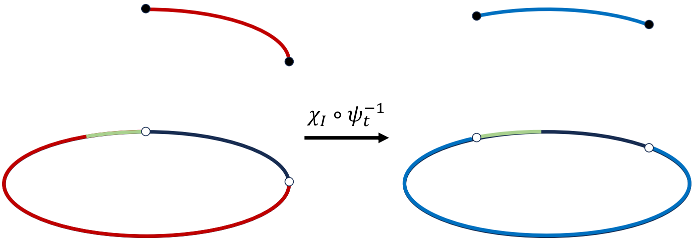
The Problem

Example

Let $I = (0, \frac{\pi}{2})$ and ψ_t be rotation counterclockwise by t radians. Note that $\widetilde{\psi_t}$, defined by $\widetilde{\psi_t}(f) = f \circ \psi_t^{-1}$, is an element of $B(L^2S^1)$. We will show that $t \mapsto \widetilde{\psi_t} M_{\chi_I} \widetilde{\psi_t}^{-1}$ (where M_g denotes multiplication by g) is not continuous at $t = 0$. By applying this operator to a function, we can see that $\widetilde{\psi_t} M_{\chi_I} \widetilde{\psi_t}^{-1} = M_{\chi_I \circ \psi_t^{-1}}$. We also have that the operator norm of M_g is the supremum norm of g.



The Problem



The left indicator function is off on the green interval, and the right indicator function is on.

For any t near 0, $\|\chi_I \circ \psi_t^{-1} - \chi_I\|_\infty = 1$.

f-Continuity

- We want to have a family of operators that is locally continuous in the norm topology under ANY trivialization.
- Trivializations of the fiber bundle can be related to one another by a smooth family of diffeomorphisms of M.
- Note that for any diffeomorphism ψ of M, we obtain an element $\widetilde{\psi}$ in $B(L^2M)$ defined by $\widetilde{\psi}(f) = f \circ \psi^{-1}$.

Definition

An operator T in $B(L^2M)$ is called f-continuous if, for any smooth family of diffeomorphisms ψ_t indexed by $\mathbb{R}$, the map $t \mapsto \widetilde{\psi_t} T \widetilde{\psi_t}^{-1}$ is continuous with respect to the operator norm.

Proposition

Let M_g in $B(L^2M)$ be multiplication by g in L^2M. Then M_g is f-continuous if and only if g is continuous.

Why do we Index by $\mathbb{R}$?

Theorem

If T is an f-continuous operator, then the map $t \mapsto \widetilde{\psi_t} T \widetilde{\psi_t}^{-1}$ is norm continuous for any smooth family of diffeomorphisms indexed by $\mathbb{R}^n$.

Remark

Because continuity is a local property, this means we can also index by a manifold.

f-Continuity

For a trivialization $\psi : \pi^{-1}(U) \to U \times M$ and an element $y \in U$, we obtain a diffeomorphism $\psi_y : \pi^{-1}(y) \to M$.

Lemma

Let $Y \to Z \xrightarrow{\pi} M$ be a smooth fiber bundle and $T = \{T_y\}$ be a family of f-continuous operators. If the map $U \to B(L^2M)$ that sends $y \mapsto \widetilde{\psi_y} T_y \widetilde{\psi_y}^{-1}$ is norm continuous for a particular trivialization $\psi : \pi^{-1}(U) \to U \times M$, then the corresponding map is norm continuous for any other trivialization.

Key Detail: If ψ and ϕ are two different trivializations, then

$$\widetilde{\phi_y} T_y \widetilde{\phi_y}^{-1} = (\widetilde{\phi_y}\widetilde{\psi_y}^{-1})(\widetilde{\psi_y} T_y \widetilde{\psi_y}^{-1})(\widetilde{\psi_y}\widetilde{\phi_y}^{-1}).$$

Note that $\phi_y \circ \psi_y^{-1}$ is a diffeomorphism of M.

The C^*-Algebra of f-Continuous Operators

Theorem

The set of all f-continuous operators in $B(L^2M)$ form a C^*-subalgebra. We will denote this C^*-algebra as $C^*(M)$.

Proposition

Let M_g in $B(L^2M)$ be multiplication by g in L^2M. Then M_g is in $C^*(M)$ if and only if g is continuous.

Proposition

The set of compact operators, $\mathcal{K}(L^2M)$, is a subset of $C^*(M)$.

The C^*-Algebra of f-Continuous Operators

Theorem

If $T \in B(L^2M)$ is an operator such that

1. $T(C^\infty(M)) \subseteq C^\infty(M)$ *and*
2. $[T, X]$ *extends to a bounded linear operator on* L^2M *for any vector field* X *on* M,

then T *is an element of* $C^*(M)$.

Corollary

The set of all bounded pseudodifferential operators of order 0, Ψ^0, *is a subset of* $C^*(M)$.

The Index and K-Theory

Definition

A Fredholm operator T in $B(\mathcal{H})$ is an operator such that $kerT$ and $kerT^*$ are finite-dimensional. The index of T is $dim(kerT) - dim(kerT^*)$. $\mathcal{F}(\mathcal{H})$ denotes the set of Fredholm operators.

Definition

Let ϵ^n be the trivial complex vector bundle of rank n over Y. We can define an equivalence relation $\sim$ over complex vector bundles as follows: $E_1 \sim E_2$ if there are n, m such that $E_1 \oplus \epsilon^n \cong E_2 \oplus \epsilon^m$. The set of all vector bundles under this equivalence relation is called $K(Y)$, or the K-Theory of Y.

The Families Index for f-Continuous Operators

Definition

For a smooth fiber bundle $M \to Z \xrightarrow{\pi} Y$ and a family T, we will say T is an f-continuous family if:

1. all the operators T_y are f-continuous and
2. the map $y \mapsto \widetilde{\psi_y} T_y \widetilde{\psi_y}^{-1}$ is norm continuous for all trivializations.

In the exceptional case where $kerT_y$ has the same dimension for every $y \in Y$, we can form a vector bundle $KerT$ out of the kernels.

Definition

For an f-continuous family T with constant dimensional kernels, the index of the family T is $[KerT] - [KerT^*]$.

What's Next?

- What exactly is this C^*-algebra?
- How do we understand the families index in relation to C^*-algebraic K-Theory and KK-Theory?

References I

M. Atiyah and D. Anderson.
K-Theory.
W.A. Benjamin, INC., 1967.

M. Atiyah and I. Singer.
The index of elliptic operators: IV.
Annals of Mathematics, 93(1):119–129, 1971.

J. Dixmier.
$C^* - Algebras$.
North-Holland Publishing Company, 1977.

A. Hatcher.
Vector Bundles and K-Theory.
Allen Hatcher, 2017.

References II

A. Kriegl and P. W. Michor.
The Convenient Setting of Global Analysis.
American Mathematical Society, 1997.

D. P. Williams.
private communication, 2023.