THE APPEL-HUMBERT THEOREM

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Abstract. This is a proof of the theorem, as can be found in [2], using some of the more modern exposition and notation, as can be found in [1].

1. Proof

1.1. **Introduction.** The goal of this short note is to present a proof of the following theorem:

Theorem 1.1.1 (Appel-Humbert, [2, p. 20])**.** *Let $X = \mathbb{C}^n/\Lambda$ be a complex torus. Then any line bundle $\mathscr{L}$ on X is of the form $\mathscr{L}(\psi, \alpha)$ where ψ is a Riemann form, and α is a semi-character with respect to ψ. Furthermore, ψ, α are uniquely determined.*

The proof is based on Mumford.

1.2. **Notation and definitions.** Throughout, we let V be a finite dimensional complex vector space of dimension n, and $\Lambda \subseteq V$ is a lattice, i.e. a subgroup of rank $2n$ such that $\Lambda \otimes \mathbb{R} \cong V$ under the canonical map $\alpha \otimes \lambda \mapsto \alpha\lambda$.

Definition 1.2.1. A *complex torus* is a complex Lie group isomorphic to $X = V/\Lambda$.

Definition 1.2.2. A *Riemann form* on V with respect to Λ is a Hermitian form $\psi : V \times V \to \mathbb{C}$ such that $\operatorname{Im} \psi(\Lambda \times \Lambda) \subseteq \mathbb{Z}$

Definition 1.2.3. A *semi-character* with respect to a Riemann form ψ is a map

$$\alpha : \Lambda \to U_1(\mathbb{R}) = \{z \in \mathbb{C} : |z| = 1\}$$

such that

$$\alpha(\lambda_1 + \lambda_2) = \alpha(\lambda_1)\alpha(\lambda_2)e^{\pi i \operatorname{Im} \psi(\lambda_1, \lambda_2)}$$

Remark 1.2.4. These could be obtained if one tries to find linear solutions when solving for the factor of automorphy given ψ.

Definition 1.2.5. The *factor of automorphy* corresponding to the pair (ψ, α) consisting of a Riemann form and a semi-character is the element $j_{(\psi,\alpha)} \in H^1(\Lambda, \mathscr{O}^\times(V))$ represented by the cocycle

$$j_{(\psi,\alpha)}(\lambda)(v) = \alpha(\lambda)e^{\pi\psi(v,\lambda)+\frac{\pi}{2}\psi(\lambda,\lambda)}$$

Remark 1.2.6. Note that $j_{(\psi,\alpha)}$ is indeed a cocycle, i.e. it satisfies

$$j(\lambda_1 + \lambda_2) = \lambda_2(j(\lambda_1)) \cdot j(\lambda_1)$$

which explicitly is

$$j(\lambda_1 + \lambda_2)(v) = j(\lambda_1)(\lambda_2 + v) \cdot j(\lambda_2)(v)$$

1.3. **Outline of the proof.** We are going to proceed in the following steps:

(1) Establish a canonical isomorphism $\mathrm{Pic}(X) \cong H^1(X, \mathscr{O}_X^\times) \cong H^1(\Lambda, \mathscr{O}^\times(V))$. Thus, for any (ψ, α), $j_{(\psi,\alpha)}$ defines a line bundle, which we denote by $\mathscr{L}(\psi, \alpha)$.
(2) Show commutativity of the diagram

$$\begin{array}{ccc} H^1(\Lambda, \mathscr{O}^\times(V)) & \longrightarrow & H^2(\Lambda, \mathbb{Z}) \\ \downarrow \cong & & \downarrow \cong \\ H^1(X, \mathscr{O}_X^\times) & \longrightarrow & H^2(X, \mathbb{Z}) \end{array}$$

so that the image of any line bundle in $H^2(\Lambda, \mathbb{Z}) \cong \bigwedge^2 \mathrm{Hom}(\Lambda, \mathbb{Z})$ gives us an alternating form ψ.
(3) Show that the image of a line bundle in $H^2(\Lambda, \mathbb{Z})$ is a Riemann form.
(4) Looking at the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(\Lambda, U_1(\mathbb{R})) & \longrightarrow & \{(\psi, \alpha)\} & \longrightarrow & \text{Riemann forms} & \longrightarrow & 0 \\ & & \downarrow \lambda & & \downarrow & & \downarrow \nu & & \\ 0 & \longrightarrow & \mathrm{Pic}^0 X & \longrightarrow & \mathrm{Pic}\, X & \longrightarrow & \ker(H^2(X, \mathbb{Z}) \to H^2(X, \mathscr{O}_X)) & \longrightarrow & 0 \end{array}$$

we prove that λ is an isomorphism. By the above, ν is also an isomorphism, hence the result.

1.4. **Step 1 - factors of automorphy.**

Proposition 1.4.1. *There exist a canonical isomorphism* $H^1(X, \mathscr{O}_X^\times) \cong H^1(\Lambda, \mathscr{O}^\times(V))$.

Proof. Let $\pi : V \to X$ be the natural projection, and let $\mathscr{L}$ be a line bundle on X. Because V is simply connected, every line bundle on it is trivial, and therefore $\pi^\star \mathscr{L} \cong V \times \mathbb{C}$. Let T_λ be translation by λ map on V given by $T_\lambda(x) = x + \lambda$. Then

$$T_\lambda^\star \pi^\star \mathscr{L} = (\pi \circ T_\lambda)^\star \mathscr{L} = \pi^\star \mathscr{L}$$

But by the definition of the pullback of a line bundle, we have

$$(T_\lambda^\star(\pi^\star \mathscr{L}))_v = (\pi^\star \mathscr{L})_{v+\lambda}$$

so under the isomorphism with the trivial bundle $T_\lambda^\star(v, z) = (v + \lambda, J(z))$, but $J(z)$ is an automorphism of $\mathbb{C}$, i.e. it must be of the form $z \mapsto j(\lambda)(v) \cdot z$ for some $j(\lambda)(v) \in \mathbb{C}^\times$. Therefore $j(\lambda) : V \to \mathbb{C}^\times$ is an element of $\mathscr{O}^\times(V)$. Moreover, as $T_{\lambda_1+\lambda_2} = T_{\lambda_1} \circ T_{\lambda_2}$, we see that

$$j(\lambda_1 + \lambda_2)(v) = j(\lambda_1)(\lambda_2 + v) \cdot j(\lambda_2)(v)$$

so that in fact j is a cocycle, i.e. $j \in Z^1(\Lambda, \mathscr{O}^\times(V))$.

If we alter the trivialization, multiplying by some $f \in \mathscr{O}^\times(V)$, then j will be replaced by the cohomologous cocycle

$$j'(\lambda)(v) = j(\lambda)(v) \cdot f(v + \lambda)/f(v)$$

so that we have defined a map $H^1(X, \mathscr{O}_X^\times) \to H^1(\Lambda, \mathscr{O}^\times(V))$. Conversely, just define the line bundle as the quotient of the trivial bundle by the relation $(v, z) \sim (v + \lambda, j(\lambda)(v) \cdot z)$. $\square$

1.5. **Step 2 - the alternating form.**

Proposition 1.5.1 ([2, p. 18])**.** *The Chern class of the line bundle corresponding to* $j \in Z^1(\Lambda, \mathscr{O}^\times(V))$ *is the alternating* 2*-form on* Λ *with values in* $\mathbb{Z}$ *given by*

$$E(\lambda_1, \lambda_2) = f_{\lambda_2}(v + \lambda_1) + f_{\lambda_1}(v) - f_{\lambda_1}(v + \lambda_2) - f_{\lambda_2}(v)$$

where $j(\lambda)(v) = e^{2\pi i f_\lambda(v)}$*.*

Proof. We get from the exponential sequence

$$0 \to \mathbb{Z} \to \mathscr{O}(V) \to \mathscr{O}^\times(V) \to 0$$

the following diagram

$$\begin{array}{ccc} H^1(\Lambda, \mathscr{O}^\times(V)) & \xrightarrow{\delta} & H^2(\Lambda, \mathbb{Z}) \\ \downarrow \cong & & \downarrow \cong \\ H^1(X, \mathscr{O}_X^\times) & \longrightarrow & H^2(X, \mathbb{Z}) \end{array}$$

This diagram commutes, because the isomorphism is compatible with the connecting morphisms. (Check!)

Write $j(\lambda)(v) = e^{2\pi i f_\lambda(v)}$, then by definition of δ, we have

$$\delta([j])(\lambda_1, \lambda_2) = f_{\lambda_1}(v + \lambda_1) - f_{\lambda_1 + \lambda_2}(v) + f_{\lambda_1}(v) \in \mathbb{Z}$$

The proof will be complete with the following Lemma. ☐

Lemma 1.5.2 ([2, p. 16])**.** *The map* $A : Z^2(\Lambda, \mathbb{Z}) \to \bigwedge^2 \operatorname{Hom}(\Lambda, \mathbb{Z})$ *defined by*

$$A(F)(\lambda_1, \lambda_2) = F(\lambda_1, \lambda_2) - F(\lambda_2, \lambda_1)$$

induces an isomorphism $H^2(\Lambda, \mathbb{Z}) \cong \bigwedge^2 \operatorname{Hom}(\Lambda, \mathbb{Z})$*.*

Moreover, for any $\xi, \eta \in \operatorname{Hom}(\Lambda, \mathbb{Z}) = H^1(\Lambda, \mathbb{Z})$*, we have* $A(\xi \cup \eta) = \xi \wedge \eta$*.*

Proof. Exercise. ☐

Remark 1.5.3. The last line is needed so that the two isomorphisms we have with $H^2(X, \mathbb{Z})$ will coincide.

1.6. **Step 3 - Riemann form.** The goal of this step is to prove the following proposition.

Proposition 1.6.1 ([2, p. 18])**.** *If we extend* E *from previous proposition* $\mathbb{R}$*-linearly to a map* $E : V \times V \to \mathbb{R}$*, then* $E(ix, iy) = E(x, y)$*.*

In order to prove this proposition, we first need to prove some auxiliary results on the structure of cohomology of X. Let $T = \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$ be the complex cotangent space to X at 0. We have a natural map $\mathscr{O}_X \otimes_{\mathbb{C}} \bigwedge^p T \to \Omega^p$ sending $\alpha \in \bigwedge^p T$ to the form ω_α defined by $(\omega_\alpha)_x = dT^\star_{-x}(\alpha)$, i.e. $(\omega_\alpha)_x(v_1, v_2) = \alpha(dT_{-x}(v_1), dT_{-x}(v_2))$. This is an isomorphism (check!) so that Ω^p is globally generated, and we get $H^q(X, \Omega^p) \cong H^q(X, \mathscr{O}_X) \otimes \Lambda^p T$. We will prove

Theorem 1.6.2 ([2, p. 4]). *Let $\overline{T} = \mathrm{Hom}_{\mathbb{C}-antilinear}(V, \mathbb{C})$, then there are natural isomorphisms*

$$H^q(X, \mathscr{O}_X) \cong \bigwedge^q \overline{T}$$

for all q, hence

$$H^q(X, \Omega^p) \cong \bigwedge^p T \otimes \bigwedge^q \overline{T}$$

Proof. Denote by $\Omega^{p,q}$ the sheaf of C^∞ complex-valued differential forms of type (p, q) on X, and let $\overline{\partial}$ be the component of the exterior derivative mapping $\Omega^{p,q}$ to $\Omega^{p,q+1}$ (derivation by $\overline{z}$). The Dolbeaut resolution

$$0 \to \mathscr{O}_X \to \Omega^{0,0} \to \Omega^{0,1} \to \Omega^{0,2} \to \dots$$

defines isomorphisms

$$H^q(X, \mathscr{O}_X) \cong \frac{Z^{0,q}_{\overline{\partial}}(X)}{\overline{\partial}(\Omega^{0,q-1}(X))}$$

where $Z^{0,q}_{\overline{\partial}}(X)$ are the $\overline{\partial}$-closed forms of type $(0, q)$. We have an isomorphism

$$\phi_{p,q} : \Omega^{0,0} \otimes_{\mathbb{C}} \left(\bigwedge^p T \otimes \bigwedge^q \overline{T} \right) \to \Omega^{p,q}$$

taking $\sum f_i \otimes \alpha_i$ to $\sum f_i \omega_{\alpha_i}$ where ω_α is the translation-invariant (p, q)-form with value α at 0. Let us show that these ω_α are all closed. Since $\omega_{\alpha \wedge \beta} = \omega_\alpha \wedge \omega_\beta$, it is enough to consider degrees $(1, 0)$ and $(0, 1)$. Since $\pi : V \to X$ is a local isomorphism, it is enough to check that $d(\pi^\star(\omega_\alpha)) = 0$. Since $\alpha \in T \oplus \overline{T}$, we see that $\pi^\star(\omega_\alpha) = d\alpha$, hence $d(\pi^\star \omega_\alpha) = d^2\alpha = 0$.

The map $\phi_{0,q}$ gives an isomorphism

$$\Omega^{0,0}(X) \otimes_{\mathbb{C}} \bigwedge^q \overline{T} \to \Omega^{0,q}(X)$$

Defining a differential $\overline{\partial}$ on $\Omega^{0,0}(X) \otimes_{\mathbb{C}} \bigwedge^q \overline{T}$ by $\overline{\partial}(f \otimes \alpha) = \overline{\partial} f \wedge \alpha$, then because the ω_α are closed, the complexes $\Omega^{0,0}(X) \otimes_{\mathbb{C}} \bigwedge^\bullet \overline{T}$ and $\Omega^{0,\bullet}(X)$ are isomorphic.

Therefore $H^q(X, \mathscr{O}_X) \cong H^q\left(\Omega^{0,0}(X) \otimes_{\mathbb{C}} \bigwedge^\bullet \overline{T}\right)$.

Finally, we will show that the inclusion $i : \bigwedge^\bullet \overline{T} \to \Omega^{0,0}(X) \otimes_{\mathbb{C}} \bigwedge^\bullet \overline{T}$ induces an isomorphism on the cohomology, i.e. $\bigwedge^q \overline{T} \cong H^q\left(\Omega^{0,0}(X) \otimes_{\mathbb{C}} \bigwedge^\bullet \overline{T}\right)$.

Let μ be the measure on X induced by the Euclidean measure on V, normalized so that $\mu(X) = 1$. It induces a map $\mu_{\bigwedge^\bullet \overline{T}} := \mu \otimes 1_{\bigwedge^\bullet \overline{T}} : \Omega^{0,0}(X) \otimes \bigwedge^\bullet \overline{T} \to \bigwedge^\bullet \overline{T}$, which is $\bigwedge^\bullet \overline{T}$-linear and such that $\mu_{\bigwedge^\bullet \overline{T}} \circ i = 1_{\bigwedge^\bullet \overline{T}}$.

We will show that $i \circ \mu_{\bigwedge^\bullet \overline{T}}$ is homotopic to the identity, hence the result.

For that purpose, we will need to introduce several maps.

First, if $\lambda \in \Lambda^\star = \mathrm{Hom}(\Lambda, \mathbb{Z})$, then λ extends to an $\mathbb{R}$-linear map $\lambda : V \to \mathbb{R}$ and the function $x \mapsto e^{2\pi i \lambda(x)}$ on V is Λ-invariant, hence factors through π as $e_\lambda \circ \pi$, for some $e_\lambda : X \to \mathbb{C}$. For any $f \otimes \alpha \in \Omega^{0,0}(X) \otimes_{\mathbb{C}} \bigwedge^\bullet \overline{T}$, we may define the Fourier coefficients

$$a_\lambda(f \otimes \alpha) = \mu(e_{-\lambda} \cdot f) \cdot \alpha \in \bigwedge^\bullet \overline{T}$$

so that

$$f = \sum_{\lambda \in \Lambda^\star} e_\lambda \otimes a_\lambda(f)$$

Next, let $\overline{C} : \Lambda^\star = \mathrm{Hom}(\Lambda, \mathbb{Z}) \to \overline{T}$ be the composition

$$\Lambda^\star \to \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \subseteq \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong T \oplus \overline{T} \to \overline{T}$$

For every $0 \neq \lambda \in \Lambda^\star$, define an element $\lambda^\star \in \mathrm{Hom}_{\mathbb{C}}(\overline{T}, \mathbb{C})$ by

$$\lambda^\star(x) = \frac{\langle x, \overline{C}(\lambda)\rangle}{2\pi i \left|\overline{C}(\lambda)\right|^2}$$

Then $\lambda^\star$ induces a map $\lambda^\star : \bigwedge^p \overline{T} \to \bigwedge^{p-1} \overline{T}$ (inner multiplication / contraction by $\lambda^\star$) via

$$\lambda^\star(\alpha_1 \wedge \ldots \wedge \alpha_p) = \sum_{k=1}^{p} (-1)^{p-k} \lambda^\star(\alpha_k) \cdot \alpha_1 \wedge \ldots \wedge \widehat{\alpha}_k \wedge \ldots \wedge \alpha_p$$

Now, for any $\omega \in \Omega^{0,0}(X) \otimes \bigwedge^p \overline{T}$ we define $k(\omega) \in \Omega^{0,0}(X) \otimes \bigwedge^{p-1} \overline{T}$ by

$$k(\omega) = \sum_{0 \neq \lambda \in \Lambda^\star} (-1)^{p-1} e_\lambda \otimes \lambda^\star(a_\lambda(\omega))$$

This indeed defines an element (check the rate of decay of the coefficients!) uniquely by Fourier analysis, and we claim

$$\overline{\partial} k + k \overline{\partial} = 1_{\Omega^{0,0}(X) \otimes \bigwedge^\bullet \overline{T}} - i \circ \mu_{\bigwedge^\bullet \overline{T}}$$

This is verified by computing Fourier coefficients on both sides - here it is for $\lambda \neq 0$:

$$\begin{aligned} a_\lambda(\overline{\partial} k \omega + k \overline{\partial} \omega) &= (-1)^p \left(2\pi i a_\lambda(k\omega) \wedge \overline{C}(\lambda) + \lambda^\star(a_\lambda(\overline{\partial}\omega))\right) \\ &= 2\pi i \left(\lambda^\star(a_\lambda(\omega)) \wedge \overline{C}(\lambda) + \lambda^\star(a_\lambda(\omega) \wedge \overline{C}(\lambda))\right) \\ &= a_\lambda(\omega) \end{aligned}$$

where we used that $a_\lambda(\overline{\partial}\omega) = (-1)^p 2\pi i \left(a_\lambda(\omega) \wedge \overline{C}(\lambda)\right)$ and $2\pi i \lambda^\star(\overline{C}(\lambda)) = 1$. □

The proof of theorem actually gives us more corollaries.

Corollary 1.6.3. *The following diagram commutes:*

$$\begin{array}{ccc} H^p(X, \mathscr{O}_X) \times H^q(X, \mathscr{O}_X) & \xrightarrow{\cup} & H^{p+q}(X, \mathscr{O}_X) \\ \downarrow \cong & & \downarrow \cong \\ \bigwedge^p \overline{T} \times \bigwedge^q \overline{T} & \xrightarrow{\wedge} & \bigwedge^{p+q} \overline{T} \end{array}$$

Corollary 1.6.4. *The natural map induced by cup product*

$$\bigwedge^q H^1(X, \mathscr{O}_X) \to H^q(X, \mathscr{O}_X)$$

is an isomorphism.

A similar computation with the cohomology of the de-Rham complex

$$0 \to \mathbb{C} \to \Omega^0 \to \Omega^1 \to \dots$$

shows that $H^n(X,\mathbb{C}) \cong \bigwedge^n \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C})$, in a way that cup products are compatible with exterior products. Also, since $\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) = T \oplus \overline{T}$, this shows the Hodge decomposition:

$$H^n(X,\mathbb{C}) \cong \bigoplus_{p+q=n} H^q(X,\Omega^p)$$

Next, we would like to establish a certain compatibility between the maps we have:

Proposition 1.6.5. *The following diagram commutes:*

$$\begin{array}{ccccc} H^1(X,\mathbb{Z}) & \xrightarrow{\alpha} & H^1(X,\mathbb{C}) & \xrightarrow{\beta} & H^1(X,\mathscr{O}_X) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ \Lambda^\star = \mathrm{Hom}(\Lambda,\mathbb{Z}) & \xrightarrow{1\otimes_{\mathbb{Z}}\mathbb{R}} & \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) = T\oplus\overline{T} & \xrightarrow{pr_2} & \overline{T} \end{array}$$

Proof. Let $p_{0,1} : \Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1} \to \Omega^{0,1}$ be the natural projection. Then the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{C} & \to & \Omega^0 & \xrightarrow{d} & \ker(\Omega^1 \to \Omega^2) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow p_{0,1} & & \\ 0 & \to & \mathscr{O}_X & \to & \Omega^{0,0} & \to & \ker(\Omega^{0,1} \to \Omega^{0,2}) & \to & 0 \end{array}$$

Passing to cohomology, we obtain the commutative diagram

$$\begin{array}{ccccc} T\oplus\overline{T} & \to & H^0(X,\ker(\Omega^1 \to \Omega^2)) & \xrightarrow{\delta} & H^1(X,\mathbb{C}) \\ \downarrow & & \downarrow p_{0,1} & & \downarrow \beta \\ \overline{T} & \to & H^0(X,\ker(\Omega^{0,1} \to \Omega^{0,2})) & \xrightarrow{\delta} & H^1(X,\mathscr{O}_X) \end{array}$$

establishing commutativity of the right square.

For the left square, note that the isomorphism on the left is given as follows - for $a \in H^1(X,\mathbb{Z})$, if $\phi : S^1 \to X$ is a loop in X representing $[\phi] \in \pi_1(X)$, then $\phi^\star(a) \in H^1(S^1,\mathbb{Z}) \cong \mathbb{Z}$. Let us denote the canonical isomorphism on the right by $\epsilon : H^1(S^1,\mathbb{Z}) \to \mathbb{Z}$. This way, one obtains $\tilde{a} : \pi_1(X) \to \mathbb{Z}$ defined by

$$\tilde{a}([\phi]) = \epsilon(\phi^\star(a))$$

In particular, for $\lambda \in \Lambda$, let $\phi_\lambda : S^1 \to X$ be the loop

$$\phi_\lambda(t) = \pi(t\lambda) \quad t \in S^1 = \mathbb{R}/\mathbb{Z}$$

Then a determines $\tilde{a} \in \Lambda^\star$ by $\tilde{a}(\lambda) = \epsilon(\phi_\lambda^\star(a))$. Consider $\alpha(a) \in H^1(X,\mathbb{C})$. There is a unique $b \in T \oplus \overline{T}$ s.t. if $\omega_b \in H^0(X,\ker(\Omega^1 \to \Omega^2))$ is the invariant 1-form on X with value b at

0, $\delta(\omega(b)) = \alpha(a)$. (By definition of the middle isomorphism). We pull back to S^1 to get $\delta(\phi_\lambda^\star(\omega_b)) = \phi_\lambda^\star(\alpha(a)) = \alpha(\phi_\lambda^\star(a))$. In S^1, we have $\epsilon(\delta(\eta)) = \int_{S^1} \eta$, hence

$$\begin{aligned}\tilde{a}(\lambda) &= \epsilon(\phi_\lambda^\star(a)) \\ &= \epsilon(\delta(\phi_\lambda^\star(\omega_b))) \\ &= \int_{S^1} \phi_\lambda^\star(\omega_b) \\ &= \int_0^\lambda \pi^\star(\omega_b) \\ &= b(\lambda)\end{aligned}$$

so that $\tilde{a}$ is simply the restriction of b to Λ. □

Furthermore, compatibility with cup products gives us the following:

Corollary 1.6.6. *The following diagram commutes*

$$\begin{array}{ccccc} H^n(X,\mathbb{Z}) & \longrightarrow & H^n(X,\mathbb{C}) & \longrightarrow & H^n(X,\mathscr{O}_X) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ \bigwedge^n(\Lambda^\star) & \xrightarrow{1\otimes_{\mathbb{Z}}\mathbb{R}} & \bigwedge^n(T\oplus\overline{T}) & \xrightarrow{pr_{0,n}} & \bigwedge^n(\overline{T}) \end{array}$$

We can now prove proposition 1.6.1.

Proof. (of proposition 1.6.1). Consider the commutative diagram

$$\begin{array}{ccccccc} H^1(X,\mathscr{O}_X) & \longrightarrow & H^2(X,\mathbb{Z}) & \longrightarrow & H^2(X,\mathbb{C}) & \longrightarrow & H^2(X,\mathscr{O}_X) \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ & & \bigwedge^2 \operatorname{Hom}(\Lambda,\mathbb{Z}) & \xrightarrow{i} & \bigwedge^2(T\oplus\overline{T}) & \xrightarrow{j} & \bigwedge^2\overline{T} \end{array}$$

Since E is in the image of the leftmost map, by the exactness of the exponential sequence, $(j\circ i)(E) = 0$. Since $i(E)$ is the $\mathbb{R}$-linear extension of E, we denote it again by E, and write $E = E_1 + E_2 + E_3$, with $E_1 \in \bigwedge^2 T$, $E_2 \in \bigwedge^2 \overline{T}$ and $E_3 \in T \oplus \overline{T}$. Since E is real, it is fixed by conjugation, and it follows that $E_1 = \overline{E_2}$. Now, j is the projection onto the second factor, so that $j(E) = E_2$. But $j(E) = 0$, hence $E = E_3$. Therefore

$$E(ix, iy) = i \cdot (-i) \cdot E(x,y) = E(x,y)$$

□

1.7. **Step 4 - Finish the proof.**

Proof. (of Theorem 1.1.1). Let ψ be a Riemann form, α a semi-character with respect to ψ, and $j_{(\psi,\alpha)} \in H^1(\Lambda, \mathscr{O}^\times(V))$ the corresponding factor of automorphy. Denote by $\mathscr{L}(\psi,\alpha) \in H^1(X, \mathscr{O}_X^\times)$ the corresponding line bundle.

Note that $\mathscr{L}(\psi_1,\alpha_1) \otimes \mathscr{L}(\psi_2,\alpha_2)$ will have factor of automorphy $j_{(\psi_1,\alpha_1)} \cdot j_{(\psi_2,\alpha_2)}$, and finally that

$$j_{(\psi_1,\alpha_1)} \cdot j_{(\psi_2,\alpha_2)} = j_{(\psi_1+\psi_2,\alpha_1\alpha_2)}$$

Therefore, if we denote by $\mathscr{G}$ the group of pairs $\{(\psi, \alpha)\}$ such that ψ is a Riemann form for Λ and α a semi-character with respect to ψ, with multiplication defined by

$$(\psi_1, \alpha_1) \cdot (\psi_2, \alpha_2) = (\psi_1 + \psi_2, \alpha_1 \alpha_2)$$

then the map $\beta : \mathscr{G} \to \operatorname{Pic} X$ is a group homomorphism. If we denote by $\mathscr{G}_1 = \operatorname{Hom}(\Lambda, U_1(\mathbb{R})) = \{(0, \alpha) \in \mathscr{G}\}$ the subgroup of semi-characters with respect to the zero Riemann form, and by $\mathscr{G}_2$ the group of Riemann forms with respect to Λ, then we have a commutative diagram with exact rows [1]

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{G}_1 & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{G}_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta} & & \downarrow & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & \operatorname{Pic}^0 X & \longrightarrow & \operatorname{Pic} X & \longrightarrow & \ker(H^2(X, \mathbb{Z}) \to H^2(X, \mathscr{O}_X)) & \longrightarrow & 0
\end{array}$$

Indeed, for $\psi \in \mathscr{G}_2$, let $\gamma(\psi) = \operatorname{Im} \psi|_\Lambda$. Since ψ is Hermitian, $\gamma(\psi)(ix, iy) = \gamma(\psi)(x, y)$ and as ψ is a Riemann form, we see that $\gamma(\psi) \in \bigwedge^2 \operatorname{Hom}(\Lambda, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$. Further, from the proof of proposition 1.6.1, we see that $j(\gamma(\psi)) = 0$, hence $\gamma(\psi) \in \ker(H^2(X, \mathbb{Z}) \to H^2(X, \mathscr{O}_X))$.

By proposition 1.6.1, for any element $E \in \ker(H^2(X, \mathbb{Z}) \to H^2(X, \mathscr{O}_X))$, we have $E(ix, iy) = E(x, y)$, hence the form $\psi(x, y) = E(ix, y) + iE(x, y)$ is Hermitian, so $\psi \in \mathscr{G}_2$, and the map γ is surjective, hence an isomorphism.

Thus, to show that the middle map is an isomorphism, it is enough to show that β is an isomorphism. Let $\alpha \in \operatorname{Hom}(\Lambda, U_1(\mathbb{R}))$ be such that $\beta(\alpha) = \mathscr{O}_X$. Note that the factor of automorphy is then $j_{(0,\alpha)}(v) = \alpha(\lambda)$. Since this is the trivial line bundle, there exists $g \in \mathscr{O}^\times(V)$ such that

$$\alpha(\lambda) = \frac{g(v + \lambda)}{g(v)}$$

Let K be the fundamental parallelogram for X, so that $K + \Lambda = V$ and K is compact. Then we see that

$$|g(v)| = |g(k + \lambda)| = |\alpha(\lambda) g(k)| \leq \sup_K |g(k)|$$

because $|\alpha(\lambda)| = 1$. But then g is bounded, hence constant, and we see that $\alpha(\lambda) = 1$, hence β is injective.

Consider the commutative diagram

$$\begin{array}{ccccc}
H^1(\Lambda, \mathbb{C}) & \longrightarrow & H^1(\Lambda, \mathscr{O}(V)) & \longrightarrow & \ker(H^1(\Lambda, \mathscr{O}^\times(V)) \to H^2(\Lambda, \mathbb{Z})) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
H^1(X, \mathbb{C}) & \longrightarrow & H^1(X, \mathscr{O}_X) & \longrightarrow & \ker(H^1(X, \mathscr{O}_X^\times) \to H^2(X, \mathbb{Z})) = \operatorname{Pic}^0 X
\end{array}$$

By proposition 1.6.5, the map $H^1(X, \mathbb{C}) \to H^1(X, \mathscr{O}_X)$ is surjective. Therefore, since the exponential maps on the right are surjective, for any line bundle $\mathscr{L} \in \operatorname{Pic}^0 X$, there exists some factor of automorphy $j \in H^1(\Lambda, \mathbb{C})$ giving rise to it. But that means that $j(\lambda)(v) = \alpha(\lambda)$ for all v, for some $\alpha \in \operatorname{Hom}(\Lambda, \mathbb{C}^\times)$. Write $\alpha(\lambda) = e^{2\pi\gamma_\lambda}$, then

$$\gamma_{\lambda_1 + \lambda_2} - \gamma_{\lambda_1} - \gamma_{\lambda_2} \in i\mathbb{Z}$$

[1] Actually we haven't shown that $\mathscr{G} \to \mathscr{G}_2$ is surjective, but this is a fun exercise. Hint: Look for linear solutions for the exponents.

Then $\operatorname{Re}\gamma_\lambda$ is additive and can be extended to an $\mathbb{R}$-linear map $l : V \to \mathbb{R}$. Let $L : V \to \mathbb{C}$ be defined by $L(v) = l(v) - il(iv)$, so that L is $\mathbb{C}$-linear and $\operatorname{Re} L = l$. Now, the function $f(v) = e^{-2\pi L(v)}$ lies in $\mathscr{O}^\times(V)$, hence the factor of automorphy $j(\lambda)(v) = \alpha(\lambda) \cdot \frac{f(v+\lambda)}{f(v)}$ is cohomologous to α. But

$$\alpha(\lambda) \cdot \frac{f(v+\lambda)}{f(v)} = \alpha(\lambda) \cdot \frac{e^{-2\pi L(v+\lambda)}}{e^{-2\pi L(v)}} = e^{2\pi\gamma_\lambda} \cdot e^{-2\pi L(\lambda)} = e^{2\pi(\gamma_\lambda - L(\lambda))}$$

Note that by choice of L, we have

$$\operatorname{Re}(\gamma_\lambda - L(\lambda)) = 0$$

so that $j(\lambda)(v) \in U_1(\mathbb{R})$, showing that β is surjective, and finishing the proof. □

References

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