Dimensions of cusp spaces of Hilbert Modular Forms

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Dimension formula

Theorem (vdG, Theorem. IV.4.4)

Let Γ be a Hilbert modular group. The dimension of the space of cusp forms of even weight $2m \geq 4$ on Γ is

$$\dim S_{2m}(\Gamma) = \frac{(2m-1)^2}{4} vol(\Gamma \backslash \mathfrak{h}^2) + \sum \sum_{\substack{\zeta^n = 1 \\ \zeta \neq 1}} \frac{\zeta^{(q+1)m}}{n(1-\zeta)(1-\zeta^q)} + \sum_{\sigma} \chi_{\sigma}(M_{\sigma}, V_{\sigma}),$$

where the first sum is over all quotient singularities of Y_Γ with type $(n; q, 1)$, and the last sum extends over all cusps of Γ.

Dimension formula: torsion-free case

Proposition (vdG, Prop. IV.4.1)

Let Γ be a torsion-free Hilbert modular group. The dimension of the space of cusp forms of even weight $k \geq 4$ on Γ is

$$\dim S_k(\Gamma) = \frac{(k-1)^2}{4} vol(\Gamma\backslash\mathfrak{h}^2) + \sum_{\sigma} \chi_\sigma(M_\sigma, V_\sigma),$$

where the sum extends over all cusps of Γ.

Right derived functors

Given a left-exact functor $F : \mathbf{A} \to \mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are "nice enough", we can define a right derived functor as follows:
Let $A \in \mathbf{A}$, we construct an injective resolution

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Take cohomology, and define the i^{th} *right derived functor* $R^i F(A) := H^i(F(I^\bullet))$.

Some sheaf cohomology

Let X be a topological space, $\mathbf{Sh}(X)$ the category of sheaves of abelian groups on X, $\mathbf{Ab}$ the category of abelian groups. Apply the previous setting to the functor of global sections

$$\begin{aligned} \Gamma(X,-):\ \mathbf{Sh}(X) &\to \mathbf{Ab} \\ \mathcal{F} &\mapsto \mathcal{F}(X) \end{aligned}$$

The cohomology functor $H^i(X,-)$ is the ith right derived functor of $\Gamma(X,-)$, $i \geq 0$. For any sheaf $\mathcal{F}$, the groups $H^i(X,\mathcal{F})$ are the cohomology groups of $\mathcal{F}$.

Relations to other cohomology theories

- Singular: let X be a topological space, $\underline{\mathbb{Z}}$ the constant sheaf,

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- De Rham: let M be a smooth manifold, $\underline{\mathbb{R}}$ the constant sheaf,

$$H^n_{dR}(M) \cong H^n(M,\underline{\mathbb{R}})$$

The Euler characteristic of a line bundle

Let $\mathcal{F}$ be a sheaf of vector spaces on a topological space X. Then $H^i(X, \mathcal{F})$ inherits a natural vector space structure. The Euler characteristic of $\mathcal{F}$ is defined as

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Let L be a line bundle; the sections of L give rise to a sheaf $\mathcal{L}$, and we denote

$$\chi(L) = \chi(X, L) \coloneqq \chi(X, \mathcal{L}).$$

Proof of Prop. IV.4.1

Recall: The Hilbert modular forms of weight $2m$ on Γ can be viewed as sections over Y_Γ of $L^{\otimes m}$, where $L = \Omega^2(\log D)$ and $D = \sum S_i$.

$$\begin{aligned}\chi(L^{\otimes m}) &= \frac{c_1(L^{\otimes m})(c_1(L^{\otimes m}) + c_1(Y_\Gamma))}{2} + \chi(Y_\Gamma) \\ &= \frac{mc_1(L)(mc_1(L) + c_1(Y_\Gamma))}{2} + \chi(Y_\Gamma)\end{aligned}$$

The above follows from Riemann-Roch, and since $L^{\otimes m}$ is a line bundle:

$$c_2(L^{\otimes m}) = 0, \qquad c_1(L^{\otimes m}) = mc_1(L)$$

Proof (cont.)

Also, $c_1(L) = -c_1(Y_\Gamma) + \sum s_i$, where the s_i are Poincaré duals of the homology classes of the S_i.

$$\begin{aligned}\chi(L^{\otimes m}) &= \frac{mc_1(L)(mc_1(L) + c_1(Y_\Gamma))}{2} + \chi(Y_\Gamma)\\ &= \frac{mc_1(L)((m-1)c_1(L) + \sum s_i)}{2} + \chi(Y_\Gamma)\\ &= \frac{m(m-1)c_1(L)^2}{2} + \chi(Y_\Gamma)\end{aligned}$$

The latter equality follows from $c_1(L) \cdot s_i = 0$.

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The latter equality follows from $c_1(L) \cdot s_i = 0$.
Recall: $c_1(L)^2 = 2\mathrm{vol}(\Gamma \backslash \mathfrak{h}^2)$, so

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By Serre duality, $\chi(L^{\otimes m} \otimes \Omega^2_{Y_\Gamma}) = \chi(L^{-m})$, so

$$\chi(L^{\otimes m} \otimes \Omega^2_{Y_\Gamma}) = \chi(L^{-m}) = m(m+1)\mathrm{vol}(\Gamma\backslash\mathfrak{h}^2) + \chi(Y_\Gamma).$$

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Recall: $\chi(\mathcal{F}) = \sum(-1)^i \dim H^i(X, \mathcal{F})$.
Kodaira's vanishing theorem gives conditions when sheaf cohomology groups are trivial:

$$H^q(X, \mathcal{F} \otimes \Omega^p_X) = 0, \quad p + q > d,$$

where X is a smooth projective scheme of dimension d, $\mathcal{F}$ is an ample invertible sheaf.

Proof (cont.)

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Therefore, $\dim H^i(Y_\Gamma, L^{\otimes m} \otimes \Omega^2_{Y_\Gamma}) = 0$ when $i > 0, m \geq 1$, and

$$\chi(L^{\otimes(m-1)} \otimes \Omega^2_{Y_\Gamma}) = \dim H^0(Y_\Gamma, L^{\otimes(m-1)} \otimes \Omega^2_{Y_\Gamma})$$

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Recall: $\chi(Y_\Gamma) = \frac{1}{4}\mathrm{vol}(\Gamma\backslash\mathfrak{h}^2) + \sum_\sigma \chi(M_\sigma, V_\sigma)$.
Putting it all together, we get the result.

Example (for B-side?)

Try to find dimensions of cusp form spaces for $\Gamma = \Gamma(2)$ and $K = \mathbb{Q}(\sqrt{13})$.

A motivating example

Recall: $\chi(X, \mathcal{F}) = \sum_i (-1)^i h^i(X, \mathcal{F})$
Let $X = \mathbb{P}^n$, $\mathcal{F} = \mathcal{O}(m)$. Then for $m \geq 0$,

$$h^0(\mathbb{P}^n, \mathcal{O}(m)) = \binom{n+m}{m} = \frac{(m+1)(m+2)\dots(m+n)}{n!},$$

a polynomial in m.
Equality not true for all m (e.g. for $m \leq -n-1$). However, it is always true that

$$\chi(\mathbb{P}^n, \mathcal{O}(m)) = \frac{(m+1)(m+2)\dots(m+n)}{n!}.$$

The Hilbert function

Let $\mathcal{F}$ be a coherent sheaf over a projective k-scheme X. The **Hilbert function** of $\mathcal{F}$ is defined as

$$h_{\mathcal{F}}(m) := h^0(X, \mathcal{F}(m)).$$

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Theorem (Vakil, 18.6.1)

For $m \gg 0$, $h^0(X, \mathcal{F}(m))$ agrees with a polynomial $p_{\mathcal{F}}(m)$ of degree $\dim Supp\mathcal{F}$. In particular, for $m \gg 0$, $h^0(X, \mathcal{O}(m))$ is a polynomial with degree $\dim X$.

Call $p_{\mathcal{F}}(m)$ the **Hilbert polynomial**; write $p_X(m) := p_{\mathcal{O}_X}(m)$.

Examples

- $p_{\mathbb{P}^n}(m) = \binom{n+m}{m}$
- Let $\iota : H \hookrightarrow \mathbb{P}^n$ a degree d hypersurface. The SES

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \iota_* \mathcal{O}_H \to 0$$

implies

$$p_H(m) = p_{\mathbb{P}^n}(m) - p_{\mathbb{P}^n}(m-d) = \binom{n+m}{n} - \binom{m+n-d}{n}.$$

Existence of a curve in $\mathbb{P}^2$ not isomorphic to $\mathbb{P}^1$

Note that $p_X(0) = \chi(X, \mathcal{O}_X) = \chi(X)$. In particular, since $p_{\mathbb{P}^1}(m) = m + 1$, we have $\chi(\mathbb{P}^1) = 1$

Nice application: We can use the Hilbert polynomials to get curves in $\mathbb{P}^2$ that are not $\mathbb{P}^1$.

Let C be a degree $d > 2$ curve in $\mathbb{P}^2$. Then

$$p_C(m) = \frac{(m+1)(m+2)}{2} - \frac{(m-d+1)(m-d+2)}{2}.$$

At $m = 0$: $p_C(0) = \frac{-d^2+3d}{2} \neq 1$, so C is not isomorphic to $\mathbb{P}^1$.