LOCALLY SYMMETRIC VARIETIES

1. Introduction

We want to generalize what we have with the modular curves - why is the modular curve so wonderful?

There we have a Hermitian symmetric domain $\mathcal{H}$, we take its quotient by a discrete group $\Gamma \subseteq SL_2(\mathbb{Z})$, to get the modular curve $Y(\Gamma) = \Gamma \backslash \mathcal{H}$.

Now, that we have this generalization, we would like to be able to do number theory with it.

One of the main advantages of the modular curve is that it can be compactified to an algebraic variety (even more than that, this variety is even defined over $\mathbb{Q}$ (even over $\mathbb{Z}[1/6]$))

Today we take one step, and show that in general, one can do this compactification (minimal compactification) in the general case and obtain an algebraic variety.

In addition, there are some properties that will be important later (maybe mention what are those and why)

There are several properties that these modular curves have:

1. They are moduli spaces for elliptic curves with some extra structure.
2. They are Riemann surfaces, and can be compactified - This allows us to consider cusps and cusp forms, using complex analysis to obtain results on elliptic curves.
3. They are algebraic curves, and in fact they are defined over a number field. This allows us to use them for number theory - questions of modularity or complex multiplication can be interpreted in this way.

Now we generalize it to other Hermitian Symmetric Domains - so far we have seen that every irreducible HSD can be identified as the connected component of the set of all Hodge structures of type $d$ on a vector space $V$, equipped with a polarization and a set of tensors. ($S(d,T)$). Ciaran will show us how this is true for our original example of $\mathcal{H}$.

We then have to consider which subgroups should be analogues of $SL_2(\mathbb{Z})$ and its congruence subgroups. For that we have to consider the above useful properties of the modular curves $\Gamma \backslash \mathcal{H}$.

1. Our varieties will be moduli spaces for variations of Hodge structures (Ciaran will show us next time why that’s also true in the modular curve case)
2. We will show that for “nice” (discrete, torsion free) subgroups these are complex manifolds, and add other condition (finite covolume) so that they can be compactified.
3. We will show that for “nice” (arithmetic, torsion-free) subgroups, they will have a (canonical) structure of an algebraic variety (still over $\mathbb{C}$).
4. We will show (that will be important later on) that $\Gamma \backslash D$ has finitely many automorphisms as a complex manifold.
2. Quotients of Hermitian Symmetric Domains by Discrete Groups

Recall the following:

**Definition 1.** Let $D$ be a connected symmetric hermitian manifold of noncompact type. We say that $D$ is a **Hermitian Symmetric Domain**.

If need further recollection - Hermitian = hermitian metric $g$ (i.e. $g$ preserves the complex structure).

Symmetric = homogeneous (Is($D, g)$ acts transitively) and at some point $p$ (hence at all), there is an involution $s_p$ having $p$ as an isolated fixed point.

Noncompact Type = the Lie group Is($D, g$) + = $\text{Hol}(D)$+ (connected component) is of noncompact type (has no nontrivial normal subgroup $G$ such that $G(\mathbb{R})$ is compact).

Notation : $\text{Hol}(D)$ are the holomorphic automorphisms of $D$ (not necessarily preserving the metric)

We would like to take the quotient of $D$ by a subgroup $\Gamma \subseteq \text{Hol}(D)^+$. In order to get good properties for this quotient, we recall some basic topology preliminaries.

2.1. Topological Preliminaries.

**Proposition 2.** ([4], Proposition 2.4) Let $G$ be a locally compact group acting on a topological space $D$ such that for one (hence every) point $p \in D$, the stabilizer $K$ of $p$ in $G$ is compact and $gK \mapsto gp : G/K \to D$ is a homeomorphism. The following conditions on a subgroup $\Gamma$ of $G$ are equivalent:

(a) For all compact subsets $A$ and $B$ of $X$, $\{ \gamma \in \Gamma \mid \gamma A \cap B \neq \emptyset \}$ is finite.

(b) $\Gamma$ is a discrete subgroup of $G$.

**Proof.** (b)$\Rightarrow$(a): First we show that if $V$ is a compact subset of $D$, then $\pi^{-1}(V)$ is compact (i.e. $\pi$ is proper).

Consider the map $\pi : G \to D$ given by $g \mapsto gp$. Write $G = \bigcup_{i \in I} U_i$ where the $U_i$ are open with compact closures.

Then, since $\pi$ is open, and $V \subseteq \bigcup_{i \in I} \pi(U_i)$, there are some $i_1, \ldots, i_n \in I$ such that $V \subseteq \bigcup_{j=1}^n \pi(U_{i_j})$.

Then

$$\pi^{-1}(V) \subseteq \bigcup_{j=1}^n U_{i_j} \cdot K \subseteq \bigcup_{j=1}^n \overline{U_{i_j}} \cdot K$$

Since the multiplication map is continuous, this is a finite union of compact subgroups, and so compact. Since $V$ was closed, $\pi^{-1}(V)$ is a closed subset of a compact set, hence compact.

Let $\gamma \in \Gamma$ be such that $B \cap \gamma A \neq \emptyset$. Then $\pi^{-1}(B) \cap \gamma \pi^{-1}(A) = \emptyset$, and so $\gamma \in \Gamma \cap \pi^{-1}(B) \cdot \pi^{-1}(A)^{-1}$, the intersection of a discrete set with a compact set, hence finite.

(a)$\Rightarrow$(b): Let $V$ be a neighbourhood of $1$ in $G$ with compact closure. For any $p \in D$,

$$\Gamma \cap V \subseteq \{ \gamma \in \Gamma \mid \gamma x \in \overline{V} \cdot x \} = \{ \gamma \in \Gamma \mid \gamma \cdot \{x\} \cap \overline{V} \cdot x \neq \emptyset \}$$

Since $\{x\}$ and $\overline{V} \cdot x$ are compact, this set is finite by assumption. Thus, $\Gamma \cap V$ is discrete, so that $1$ is an isolated point of $\Gamma$. $\square$
Therefore, we restrict ourselves to discrete subgroups \( \Gamma \). Moreover, in this case, we have some other corollaries.

**Proposition 3.** ([4], Proposition 2.5) Let \( G \) be a locally compact group acting on a topological space \( D \) such that for one (hence every) point \( p \in D \), the stabilizer \( K \) of \( p \) in \( G \) is compact and \( gK \to gp : G/K \to D \) is a homeomorphism. Let \( \Gamma \subseteq G \) be a discrete subgroup. Then:

(a) For every \( p \in D \), \( \{g \in \Gamma \mid gp = p\} \) is finite.

(b) For any \( p \in D \), there exists a neighbourhood \( U \) of \( p \) such that for \( g \in \Gamma \), \( gU \cap U = \emptyset \) unless \( gp = p \). (local homeomorphism of \( \Gamma \backslash D \) and \( D \))

(c) for any points \( p, q \in D \) not in the same \( \Gamma \)-orbit, there exist neighbourhoods \( U \) of \( p \) and \( V \) of \( q \) such that \( gU \cap V = \emptyset \) for all \( g \in \Gamma \). (\( \Gamma \backslash D \) is Hausdorff)

**Proof.** (a) Since \( \{p\} \) is compact, \( \pi^{-1}(p) \) is compact, and so \( \pi^{-1}(p) \cap \Gamma \) is finite.

(b) Let \( V \) be a compact neighbourhood of \( p \). Let \( \gamma_1, \ldots, \gamma_n \in \Gamma \) be the finitely many elements (by Proposition 2) such that \( \gamma_i V \cap V \neq \emptyset \). Let \( \gamma_1, \ldots, \gamma_s \) be (wlog) the ones fixing \( p \).

For each \( i > s \), choose disjoint neighbourhoods \( V_i \) and \( W_i \) of of \( p \) and \( \gamma_ip \), respectively, and put

\[
U = V \cap \left( \bigcap_{i > s} V_i \cap \gamma_i^{-1}W_i \right)
\]

For \( i > s \), \( \gamma_iU \subseteq W_i \), which is disjoint from \( V_i \), which contains \( U \).

(c) Choose compact neighbourhoods \( A \) of \( p \), \( B \) of \( q \) and let \( \gamma_1, \ldots, \gamma_n \) be the finitely many elements (by Proposition 2) of \( \Gamma \) such that \( \gamma_i A \cap B \neq \emptyset \). We know \( \gamma_ip \neq q \), so we can find disjoint open neighbourhoods \( U_i \) and \( V_i \) of \( \gamma_ip \) and \( q \). Take

\[
U = A \cap \left( \bigcap_{i=1}^n \gamma_i^{-1}U_i \right), \quad V = B \cap \left( \bigcap_{i=1}^n V_i \right)
\]

\( \square \)

2.2. (Torsion free) Discrete subgroups give complex manifolds. We can now specialize this to the case at hand.

**Proposition 4.** ([5], Proposition 3.1) Let \( D \) be a Hermitian Symmetric Domain, and let \( \Gamma \) be a discrete subgroup of \( \text{Hol}(D)^+ \). If \( \Gamma \) is torsion-free, then \( \Gamma \) acts freely on \( D \), and there is a unique complex structure on \( \Gamma \backslash D \) for which the quotient map \( \pi : D \to \Gamma \backslash D \) is a local isomorphism. Relative to this structure, a map \( \varphi : \Gamma \backslash D \to M \) into a complex manifold \( M \) is holomorphic iff \( \varphi \circ \pi \) is holomorphic.

**Example 5.** If \( D = \mathcal{H} \), then \( \text{Hol}(D) = \text{PSL}_2(\mathbb{R}) \) as Möbius transformations. This is also connected, hence \( \text{Hol}(D)^+ = \text{PSL}_2(\mathbb{R}) \). \( \Gamma = \text{PSL}_2(\mathbb{Z}) \) is a discrete subgroup of \( \text{PSL}_2(\mathbb{R}) \), but it is not torsion-free! (recall \( S^2 = T^3 = 1 \)). Indeed, in this case, the action is not free, and we have the elliptic points \( i, 2i, 3i \) precisely for this reason.

However, the congruence subgroups \( \Gamma(N) \) are torsion-free for \( N > 1 \) (and also \( \Gamma_1(N) \) for \( N > 3 \). Note, however, that there are infinitely many \( N \)'s for which \( \Gamma_0(N) \) has torsion).

(Why? this is a nice exercise!)
Let $\gamma \in SL_2(\mathbb{Z})$ be an element of finite order. Its eigenvalues are then roots of unity - $\zeta_n$, and the trace is $\zeta_n + \zeta_n^{-1}$ (hence $n \in \{1, 2, 3, 4, 6\}$).

This means that the characteristic polynomial of $\gamma$ is one of the following:

$$x^2 - x + 1, x^2 + 1, x^2 + x + 1, (x + 1)^2, (x - 1)^2$$

hence conjugate to one of the following:

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & \lambda \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

The last two are of infinite order, unless $\lambda = 0$, in which case we get $\pm 1$. Thus the only elements of finite order are conjugate to either $S$ or $T$.

If $\gamma \in \Gamma(N)$, then $tr(\gamma) \equiv 2 \mod N$. In particular, if $N \geq 4$ (so that $1 \neq 2 \mod N$ and $-1 \neq 2 \mod N$), it follows that $\gamma = \pm 1$, and so its image in $PSL_2(\mathbb{Z})$ is trivial.

If $N = 3$, the only other possibility is $\zeta_6$. So we are looking for $a, b, c, d \in \mathbb{Z}$ such that $(3a + 1) + (3d + 1) = -1$ and $(3a + 1)(3d + 1) - 9bc = 1$.

Thus, $a + d = -1$ and $9(ad - bc) = -3(a + d) = 3$, a contradiction.

If $N = 2$, the only other possibility is $\zeta_4$, and so we are looking for $a, b, c, d \in \mathbb{Z}$ such that $(2a + 1) + (2d + 1) = 0$ and $(2a + 1)(2d + 1) - 4bc = 1$.

Thus, $a + d = -1$ and $4(ad - bc) = -2(a + d) = 2$, a contradiction.

How about $\Gamma_1(N)$?

When $N = 3$, we get $9ad - 3bc = 3$, hence $3ad - bc = 1$, or $3a(-1 - a) - bc = 1$, hence $3a(a + 1) + bc + 1 = 0$

Why not take $a = 0$, $b = 1$, $c = -1$. Then $d = -1$, and one gets the matrix

$$\gamma = \begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix} \in \Gamma_1(3), \text{ and indeed}$$

$$\gamma^2 = \begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

What happens for $\Gamma_0(N)$? We have to find $a, b, c, d \in \mathbb{Z}$ such that $a + d \in \{\pm 1, 0\}$ and $ad - Nbc = 1$.

For example, if $-1$ is a square root modulo $N$ (that is $N$ has an even number of primes in its decomposition that are 3 mod 4 - there are infinitely many, as there are infinitely many primes that are 1 mod 4), then one can take $a$ to be such that $a^2 \equiv -1 \mod N$, take $d = -a$, $b = (a^2 + 1)/N$, and $c = -1$. Then one gets the matrix

$$\begin{pmatrix} a & (a^2 + 1)/N \\ -N & -a \end{pmatrix} \in \Gamma_0(N) \text{ which is of order } 4$$

For the proof we will need to recall two results from Carl’s talk:

**Lemma 6. ([5], Lemma 1.5)** Let $(M, g)$ be a symmetric space, and let $p \in M$. Then the subgroup $K_p$ of $Is(M, g)^+$ fixing $p$ is compact, and

$$a \cdot K_p \mapsto a \cdot p : Is(M, g)^+ / K_p \rightarrow M$$

is an isomorphism of smooth manifolds.
Proposition 7. ([5], Proposition 1.6) Let \((M, g)\) be a Hermitian symmetric domain. The inclusions \(Is(M, g) \subseteq Hol(M)\) and \(Is(M, g) \subseteq Is(M^\infty, g)\) induce equalities
\[
Is(M^\infty, g)^+ = Is(M, g)^+ = Hol(M)^+
\]
(Reminder to self: Why this is true. In general, note that \(Is(M, g)\) is much smaller than \(Is(M^\infty, g)\), e.g. for \(C^2\), this is \(U(2) \subseteq SO(4)\). If \(G = Is(M^\infty, g)^+\), \(g\) its Lie algebra, and \(s_0 \in G\) the symmetry at \(0\), then the adjoint representation gives us an involution \(s = g \mapsto s_0 g s_0 \in Aut(g)\). Let \(p = \{X \in g \mid s X = - X\}\) be the \(-1\) space. Since \(Is(M, g)\) contains all symmetries, it contains all one-parameter subgroups \(\exp t X\) for all \(X \in p\). (It contains \(s_{1/2} s_0 = \exp t X\), where \(s_1\) is the geodesic symmetry with respect to \(\exp t X\). Hence its Lie algebra contains \(p\) and \([p, p]\). But by semisimplicity, \(p + [p, p] = g\), so the Lie algebras are the same, hence so is the connected component, Similarly for the second equality. )

Proof. (of Proposition 4) Let \(\Gamma\) be a discrete subgroup of \(Hol(D)^+\). Let \(p \in D\). By Lemma 6 and Proposition 7, we know that the stabilizer
\[
K_p = \{g \in Isom(D)^+ = Hol(D)^+ \mid gp = p\}
\]
is compact and the map \(g \mapsto gp : Hol(D)^+ / K_p \to D\) is a homeomorphism.

Therefore, we can use Proposition 3. Since \(\Gamma\) is torsion-free, the finite group \(\{g \in \Gamma \mid gp = p\}\) is trivial, and so \(\Gamma\) acts freely (by definition - trivial stabilizers) on \(D\).

Let \(\pi : D \to \Gamma \backslash D\) be the quotient map. If \(p, q \in D\) are not in the same \(\Gamma\)-orbit, considering the sets \(U, V\) from (c), \(\pi(U)\) and \(\pi(V)\) are disjoint neighbourhoods of \(\pi(p), \pi(q)\), and so \(\Gamma \backslash D\) is Hausdorff.

Let \(q \in \Gamma \backslash D\), and let \(p \in \pi^{-1}(q)\). Let \(U\) be as in (b). Then \(\pi \mid_U : U \to \pi(U)\) is a homeomorphism. (b) implies injectivity, and as a quotient map it is already continuous, open and surjective. It follows that \(\Gamma \backslash D\) is a manifold.

We proceed to define a complex structure on \(\Gamma \backslash D\). We do this by determining the sheaf of holomorphic functions.

For any open \(U \subseteq \Gamma \backslash D\), let
\[
\mathcal{O}_{\Gamma \backslash D}(U) = \{f : U \to \mathbb{C} \mid f \circ \pi \in \mathcal{O}_D(\pi^{-1}(U))\}
\]
Then this is a sheaf of rings on \(\Gamma \backslash D\), for which \(\pi\) is a local isomorphism of ringed spaces. (Indeed, \(f \mapsto f \circ \pi : \mathcal{O}_{\Gamma \backslash D} \to \pi_* \mathcal{O}_D\) is a local morphism).

Therefore, it defines a complex structure on \(\Gamma \backslash D\) for which \(\pi\) is a local isomorphism of complex manifolds.

Finally, let \(\varphi : \Gamma \backslash D \to M\) be a map such that \(\varphi \circ \pi\) is holomorphic, and let \(f \in \mathcal{O}_M(U)\) for some open \(U \subseteq M\). Then, by definition
\[
(f \circ \varphi) \circ \pi = f \circ (\varphi \circ \pi) \in \mathcal{O}_D((\varphi \circ \pi)^{-1}(U)) = \mathcal{O}_D(\pi^{-1}(\varphi^{-1}(U))
\]
It follows, by definition, that \(f \circ \varphi \in \mathcal{O}_{\Gamma \backslash D}(\varphi^{-1}(U))\). It follows that \(\varphi\) is holomorphic. \(\square\)

Remark 8. Note that \(D\) is the universal covering space for \(\Gamma \backslash D\), and \(\Gamma\) is the group of covering transformations. Moreover, for any point \(p \in D\), the map
\[
g \mapsto [\pi(p \sim gp)] : \Gamma \to \pi_1(\Gamma \backslash D, \pi(p))
\]
is an isomorphism.
(This follows from the fact that $D$ is simply connected - why do we know that? There was a note, somewhere in first chapter, that “it is known” that every hermitian symmetric domain can be embedded into $\mathbb{C}^n$ as a bounded symmetric domain. !!! Might find explanation in Helgason. Anyway, these are Harish-Chandrah models - we might need them for BB compactification anyway)

3. Subgroups of Finite Covolume

Recall that we would like to be able to compactify our space. Why? well, basically because we have much better understanding of compact spaces.

We could just ask that $\Gamma \backslash D$ will be compact. This is possible, but we would like to build a richer theory. For example, this would exclude the modular curves, which are not compact.

Here is a compromise.

We would like our quotient space $\Gamma \backslash D$ to be sufficiently small, i.e. to have finite volume. (This, in turn, will allow us to compactify it)

By definition, $D$ has a Riemannian metric $g \in T^*M \otimes T^*M$, and hence a volume element $\omega \in \Omega^n(M) = \wedge^n T^*M$, given by $\sqrt{\det g}$. (Riemannian volume form).

Since $g$ is $\Gamma$-invariant, so is $\omega$, and so it induces a volume form on the quotient. Thus, our condition is $\int_{\Gamma \backslash D} \omega < \infty$.

Example 9. Let us return to our running example, $D = \mathcal{H}$, $\Gamma = \text{PSL}_2(\mathbb{Z})$. Then $F = \left\{ z \in \mathcal{H} \mid -\frac{1}{2} < \Re(z) < \frac{1}{2}, |z| > 1 \right\}$ is a fundamental domain for $\Gamma$, the Riemannian metric is given by $\frac{(dx)^2 + (dy)^2}{y^2}$, i.e.

$$g(x, y) = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$$

It follows that $\det g(x, y) = y^{-4} \cdot dx dy$, so that $\sqrt{\det g(x, y)} = y^{-2} dx dy$. Thus

$$\int_{\Gamma \backslash D} \omega = \int_F \frac{dx dy}{y^2} \leq \int_{\frac{1}{2}}^\infty \int_{-\frac{1}{2}}^\frac{1}{2} \frac{dx dy}{y^2} = \int_{\frac{1}{2}}^\infty \frac{dy}{y^2} = 2 \sqrt{3} < \infty$$

On the other hand, quotienting out only by translations, yields infinite volume.

We now have the following Lemma.

**Lemma 10.** ([6], 1.3, Exercise 6) Let $G$ be a Lie group. Let $K \subseteq G$ be a compact subgroup. Let $\Gamma \subseteq G$ be a discrete subgroup, acting freely on $G/K$. Then $\Gamma \backslash G$ has finite volume if and only if $\Gamma \backslash G/K$ has finite volume.

**Proof.** Assume that $\Gamma \backslash G/K$ has finite volume. Let $\omega \in \Omega^n(\Gamma \backslash G/K)$ be the volume form. Let $\pi : \Gamma \backslash G \rightarrow \Gamma \backslash G/K$ be the quotient map. Then $\pi^* \omega \in \Omega^n(\Gamma \backslash G)$. Let $\mu \in \Omega^k(K)$ be the volume form on $K$ (This is the Haar measure). Let $\iota : K \rightarrow G$ be the natural inclusion. Then $\iota_* \mu \in \Omega^k(G)$. Then, by Fubini’s Theorem

$$\int_{\Gamma \backslash G} \pi^* \omega \wedge \iota_* \mu = \int_{\Gamma \backslash G/K} \omega \wedge \pi_* \iota_* \mu = \int_{\Gamma \backslash G/K} \omega \wedge \int_K \mu$$
However, $K$ is compact, hence has finite Haar measure $\int_K \mu = C_K < \infty$, so that

$$\int_{\Gamma \backslash G} \pi^* \omega \wedge \iota_\ast \mu \cdot \int_{\Gamma \backslash G / K} \omega < \infty$$

Note that $\pi^* \omega \wedge \iota_\ast \mu \in \Omega^{n+k}(\Gamma \backslash G)$, hence it is a volume form. □

Thus, it is enough to consider those discrete, torsion-free groups $\Gamma \subseteq \text{Hol}(D)^+$ which are of finite covolume.

4. Arithmetic Subgroups

4.1. Overview and commensurability. We now come to the crucial step. We would like our quotient $\Gamma \backslash D$ to have the structure of an algebraic variety.

That will take us on a long journey.

In a nutshell, the reason this happens for finite index subgroups of $\text{SL}_2(\mathbb{Z})$, is that $\text{SL}_2(\mathbb{Z})$ is not just a discrete (of finite covolume) subgroup of $\text{SL}_2(\mathbb{R})^+$. This group has something more - it can be interpreted as the stabilizer of the lattice $\mathbb{Z}^2 \subseteq \mathbb{Q}^2 \subseteq \mathbb{R}^2$ - an integral (vs. rational, vs. real) structure in an underlying vector space.

Apparently, this is the property that is responsible for the algebraicity of the modular curve.

Wait! But what about its finite index subgroups? Well, for this purpose, finite index difference will not matter. For that we make the following definition.

Definition 11. Let $S_1, S_2 \subseteq H$ be two subgroups of a group $H$. They are commensurable if $[S_i : S_i \cap S_2] < \infty$ for $i = 1, 2$.

Example 12. If $\mathbb{Z}\alpha, \mathbb{Z}\beta \subseteq \mathbb{R}$ are commensurable, then either $\alpha = \beta = 0$, or $\mathbb{Z}\alpha \cap \mathbb{Z}\beta \neq 0$, (and in particular $\alpha, \beta \neq 0$) so that there exist $m, n \in \mathbb{Z}$ such that $m\alpha = n\beta$, hence $\beta \in \mathbb{Q}^\times \alpha$. Conversely, if $\beta \in \mathbb{Q}^\times \alpha$, there exist such $m$ and $n$, and so

$$\mathbb{Z}\alpha \cap \mathbb{Z}\beta = \mathbb{Z}\alpha \cap \mathbb{Z}\alpha \cdot \frac{m}{n} = (n\mathbb{Z} \cap m\mathbb{Z}) \cdot \frac{\alpha}{n} = \text{lcm}(m, n) \cdot \mathbb{Z} \cdot \frac{\alpha}{n}$$

so that

$$[\mathbb{Z}\alpha : \mathbb{Z}\alpha \cap \mathbb{Z}\beta] = \frac{\text{lcm}(m, n)}{n} < \infty$$

and

$$[\mathbb{Z}\beta : \mathbb{Z}\alpha \cap \mathbb{Z}\beta] = \frac{\text{lcm}(m, n)}{m} < \infty$$

Thus, $\mathbb{Z}\alpha, \mathbb{Z}\beta \subseteq \mathbb{R}$ are commensurable iff $\beta \in \mathbb{Q}^\times \cdot \alpha$.

Remark 13. Commensurability is an equivalence relation. Indeed, if $H, H'$ are subgroups of $G$ of finite index, then $H \cap H'$ is of finite index in $H$, because $H/(H \cap H') \to G/H'$ is injective. It follows that if $H_1$ and $H_3$ are each commensurable with $H_2$, then $H_1 \cap H_2 \cap H_3$ has finite index in each of $H_1 \cap H_2$ and $H_2 \cap H_3$ (and therefore in $H_1$ and $H_3$). Thus, $H_1 \cap H_3$ has finite index in $H_1$ and $H_3$.


(a) If $\varphi : G \to H$ is a homomorphism, and $S_1, S_2 \subseteq G$ are commensurable, then $\varphi(S_1), \varphi(S_2)$ are commensurable.
Proof. (a) Let \( i \in \{1, 2\} \), and consider the map
\[
S_i/(S_i \cap S_2) \to \varphi(S_i)/\varphi(S_i \cap S_2) \to \varphi(S_i)/(\varphi(S_i) \cap \varphi(S_2))
\]
induced by \( \varphi \). The first map is clearly surjective, and since \( \varphi(S_i \cap S_2) \subseteq \varphi(S_i) \cap \varphi(S_2) \), the second map is surjective, so that
\[
[\varphi(S_i) : \varphi(S_i) \cap \varphi(S_2)] \leq [\varphi(S_i)/\varphi(S_i \cap S_2)] \leq [S_i : S_1 \cap S_2] < \infty
\]
\( \square \)

!!! Maybe say some more on commensurability here to make later arguments more elegant !!!

4.2. Arithmetic subgroups.

**Definition 15.** Let \( V \) be a vector space over \( \mathbb{Q} \). Let \( G \subseteq GL_V \) be an algebraic subgroup. A subgroup \( \Gamma \) of \( G(\mathbb{Q}) \) is **arithmetic** if there exists a lattice \( L \subseteq V \) such that \( \Gamma \) is commensurable with \( G_L := \{ g \in G(\mathbb{Q}) \mid gL = L \} \).

**Definition 16.** Let \( G \subseteq GL_n \) be an algebraic subgroup defined over \( \mathbb{Q} \). Let \( \Gamma \subseteq G \) be an arithmetic subgroup. Let \( \Gamma' \subseteq \Gamma \) be a subgroup such that there exists \( m \in \mathbb{Z}_{>0} \) for which
\[
\Gamma' = \{ x \in \Gamma \mid x \equiv 1 \mod m \}
\]
We say that \( \Gamma' \) is a **congruence subgroup** of \( \Gamma \).

**Proposition 17.** ([1], Proposition 7.12) Let \( G \subseteq GL_n \) be an algebraic subgroup defined over \( \mathbb{Q} \). Let \( \rho : G \to GL_V \) be an algebraic representation defined over \( \mathbb{Q} \). Let \( L \subseteq V \) be a lattice. Then there exists a congruence subgroup of \( G_Z := G(\mathbb{Q}) \cap GL_n(\mathbb{Z}) \) keeping \( L \) stable.

Proof. First note that \( G_Z = \{ g \in G(\mathbb{Q}) \mid g\mathbb{Z}^n = \mathbb{Z}^n \} \) is the stabilizer of the lattice \( L = \mathbb{Z}^n \). Thus, it is an arithmetic subgroup of \( G \), and so there is meaning to the proposition. Next, assume \( \dim V = N \), choose a basis for \( L - B = \{ v_1, v_2, \ldots, v_N \} \). Then this is also a basis for \( V \), identifying \( V \cong \mathbb{C}^N \) and \( GL_V \cong GL_N \). Since \( \rho \) is algebraic, for any \( g \in G \), the entries of the matrix \( [\rho(g)]_{B} \) are polynomials in the entries of \( g \) (over \( \mathbb{Q} \)). Namely, \( \rho(g)_{ij} = P_{ij}(\{g_{kl}\}) \) for all \( 1 \leq i, j \leq N \), with \( P_{ij} \in \mathbb{Q}[G] \).

It follows that there are polynomials \( Q_{ij} \in \mathbb{Q}[G] \) such that (Indeed, this is a linear change of variables)
\[
\rho(g)_{ij} - \delta_{ij} = Q_{ij}(\{g_{kl} - \delta_{kl}\})
\]
Moreover, as \( \rho(1) = 1 \), it follows that the \( Q_{ij} \) vanish at \( g = 1 \), i.e. \( g - 1 = 0 \), and hence have no constant term. Let \( m \) be the least common denominator of all the coefficients of the \( Q_{ij} \). Then \( Q_{ij}(\{g_{kl} - \delta_{kl}\}) \in \mathbb{Z} \) if \( g_{kl} - \delta_{kl} \equiv 0 \mod m \), i.e. \( g \equiv 1 \mod m \). Therefore, choosing such a \( g \), the entries of \( \rho(g) \) are integral, thus stabilizing \( L \).

**Corollary 18.** ([1], Corollary 7.13) (1) If \( \Gamma \) is an arithmetic subgroup of \( G \subseteq GL_V \), any lattice \( L' \subseteq V \) is contained in a lattice invariant under \( \Gamma \).
(2) Let \( \varphi : G \to G' \) be an isomorphism defined over \( \mathbb{Q} \), and \( \Gamma, \Gamma' \) arithmetic subgroups of \( G, G' \). Then \( \varphi(\Gamma) \) is commensurable with \( \Gamma' \).
(3) If \( \varphi : G \to G' \) is a morphism defined over \( \mathbb{Q} \) and \( \Gamma \subseteq G(\mathbb{Q}) \) is arithmetic, there exists an arithmetic subgroup \( \Gamma' \) of \( G' \) containing \( \varphi(\Gamma) \).
(4) If \( G = H \cdot N \) is a semidirect product of a closed normal subgroup \( N \) and a closed subgroup \( H \), defined over \( \mathbb{Q} \), then \( H \cdot N \) is an arithmetic subgroup of \( G \).

**Proof.** (1) Let \( L \subseteq V \) be a lattice such that \( \Gamma \) is commensurable with \( G_L \). Let \( B \) be a basis for \( L \). Then \( B \) identifies \( V \cong \mathbb{C}^n \) and \( GL_V \cong GL_n(\mathbb{C}) \) such that \( G_L \cong G_Z \).

Thus we may assume that \( \Gamma \) is commensurable with \( G_Z \). By Proposition 17, there exists a finite index subgroup \( \Gamma' \subseteq \Gamma \) stabilizing \( L' \). Therefore, the set of lattices \( \{gL'\}_{g \in \Gamma} \) is finite. Thus, \( \sum_{g \in \Gamma} gL' \) is a lattice, stable under \( \Gamma \).

Borel claims that (2) and (3) are immediate consequences of (1) - I don’t see that.

(2) We identify \( G \to GL_n \) and \( G' \to GL_{n'} \), so that \( \Gamma \), \( \Gamma' \) are commensurable with \( G_Z \), \( G'_Z \). (Choosing bases for the corresponding lattices \( L, L' \).) Then it is enough to show that \( \varphi(G_Z) \) is commensurable with \( G'_Z \). (Indeed, if this is the case, then \( \varphi(G_Z) \) is commensurable with \( \Gamma' \), and as \( \Gamma \) is commensurable with \( G_Z \), their images are commensurable by Lemma 14 (a)).

But, \( \varphi \) is defined over \( \mathbb{Q} \), so there is a common denominator to its polynomial entries, and this determines the finite index for commensurability.

(3) By proposition, there exists \( \Gamma_1 \subseteq \Gamma \) of finite index, such that \( \varphi(\Gamma_1) \) stabilizes \( \mathbb{Z}^n \). It follows that

\[
\varphi(\Gamma_1) \subseteq \varphi(G(\mathbb{Q})) \cap GL_{n'}(\mathbb{Z}) \subseteq G'(\mathbb{Q}) \cap GL_{n'}(\mathbb{Z}) = G'_Z
\]

is an arithmetic subgroup of \( G' \) containing \( \varphi(\Gamma_1) \). Let \( \Gamma' = \varphi(\Gamma) \cdot G'_Z \). Then (as sets)

\[
\Gamma'/\langle \Gamma' \cap G'_Z \rangle = \Gamma'/G'_Z = \varphi(\Gamma)/\varphi(\Gamma) \cap G'_Z
\]

and as \( \varphi(\Gamma_1) \subseteq \varphi(\Gamma) \cap G'_Z \), it follows that

\[
[\Gamma' : \Gamma' \cap G'_Z] \leq [\varphi(\Gamma) : \varphi(\Gamma_1)] \leq [\Gamma : \Gamma_1] < \infty
\]

Finally, \( G'_Z / (\Gamma' \cap G'_Z) = \{1\} \) is trivial, therefore, we see that \( \Gamma' \) is commensurable with \( G'_Z \), hence arithmetic. (Maybe 2 can be deduced similarly ?)

(4) Let \( \pi : G \to H = G/N \) be the projection. For \( g \in G \), \( \pi(g)^{-1} \cdot g \in N \). By (3), there exists a subgroup \( \Gamma \) of finite index in \( G_Z \) such that \( \pi(\Gamma) \subseteq H_Z \). Then \( \Gamma \subseteq H_Z \cdot N_Z \).

It now follows from the corollary, that one may define equivalently:

**Definition 19.** ([5]) Let \( G \) be an algebraic group over \( \mathbb{Q} \). A subgroup \( \Gamma \) of \( G(\mathbb{Q}) \) is arithmetic if it is commensurable with \( G(\mathbb{Q}) \cap GL_n(\mathbb{Z}) \) for some embedding \( G \to GL_n \). It is then commensurable with \( G(\mathbb{Q}) \cap GL_{n'}(\mathbb{Z}) \) for every embedding \( G \to GL_{n'} \).

Indeed, if \( \Gamma \) commensurable with \( G_L \) for some \( L \), write \( \varphi : GL_V \to GL_V \) for the change of basis isomorphism, to change to a basis of \( L \). Then \( \varphi(G_L) = G_Z \) is commensurable with \( G'_L \) by (2). For the second assertion, let \( \varphi : \iota_1(G) \to \iota_2(G) \) be the composition \( \iota_2 \circ \iota_1^{-1} \). This is a \( \mathbb{Q} \)-isomorphism, and so \( G_Z \) is commensurable with \( \varphi(G_Z) \).

**Example 20.** Here are some examples of arithmetic groups.

1) \( GL_n(\mathbb{Z}) \) in \( GL_n(\mathbb{R}) \).
2) Let \( F \) be a nondegenerate rational quadratic form. Its unit group \( O(F) \cap GL_n(\mathbb{Z}) \) is an arithmetic group.
3) The group $\text{Sp}(2n,\mathbb{Z}) = \{ M \in GL_{2n}(\mathbb{Z}) \mid ^tM \cdot J \cdot M = J \}$, where $J$ is the matrix
$$
\begin{pmatrix}
0 & I_n \\
-I_n & 0
\end{pmatrix}
$$

4) If $K$ is a number field with basis $\{\omega_1, \ldots, \omega_d\}$ over $\mathbb{Q}$, one can embed $K^\times \hookrightarrow GL_d(\mathbb{Q})$ by the regular representation. Thus, its image is the $\mathbb{Q}$-points of a group $G$, which is a $d$-dimensional torus defined over $\mathbb{Q}$. $G(\mathbb{C})$ is then the commutant of $K$ in $GL_d(\mathbb{C})$. If the basis consists of integral elements, then $G \cap M_d(\mathbb{Z})$ is identified with the nonzero algebraic integers of $K$. The arithmetic group $G_\mathbb{Z}$ is then the unit group $\mathcal{O}_K^\times$.

5) More generally, one may consider a finite dimensional $\mathbb{Q}$-algebra, $A$, the group $G = A^\times$, and the arithmetic group $G_L$ of units of a lattice $L$ in $A(\mathbb{Q})$:
$$
G_L := \{ g \in A \mid gL = L \}
$$

4.3. Arithmetic subgroups and compactness (for finite covolume). Next, we would like to say that, at least in our case of interest, such subgroups are of the form we would like to consider. Well, they are clearly discrete but what about finite covolume?

Before we are able to answer that, we need to recall several things.

Remark 21. The set of lattices in $\mathbb{R}^n$, $\mathfrak{R}$, can be identified canonically with $GL_n(\mathbb{R})/GL_n(\mathbb{Z})$, and as such can be equipped with the quotient topology. If $G \subseteq GL_n$ is an algebraic group, then $G(\mathbb{R})/G_\mathbb{Z}$ can be evidently embedded in $\mathfrak{R}$.

First, in order to understand when $G(\mathbb{R})/G_\mathbb{Z}$ is compact, we would like to have a good criterion for compactness in $\mathfrak{R}$.

4.3.1. Mahler’s criterion - compactness in $\mathfrak{R}$. We first recall the Iwasawa decomposition of $GL_n(\mathbb{R})$ (basically because of Graham-Schmidt):

Proposition 22. Let $A$ be the group of diagonal matrices with positive entries. Let $K, N$ denote respectively the orthogonal group and the strictly upper triangular group, formed of upper triangular matrices with all eigenvalues equal to 1. The map
$$
(k, a, n) \mapsto k \cdot a \cdot n : K \times A \times N \to GL_n(\mathbb{R})
$$

is a homeomorphism.

This allows us to consider the following.

Definition 23. We call a Siegel set of $GL_n(\mathbb{R})$ any set of the form $S_{t,u} = K \cdot A_t \cdot N_u$ (where $t, u \in \mathbb{R}_{>0}$) where
$$
A_t = \{ a \in A \mid a_{it} \leq t \cdot a_{i+1,i+1} \quad \forall i = 1, 2, \ldots, n-1 \}
$$
$$
N_u = \{ n \in N \mid |n_{ij}| \leq u \quad \forall 1 \leq i < j \leq n \}
$$

Remark 24. Note that as $N$ is a closed subgroup of $GL_n(\mathbb{R})$ (homeomorphic to $\mathbb{R}^{n(n-1)/2}$ via $\theta : n \mapsto (n_{ij})_{1 \leq i \leq j \leq n}$), $N_u$ is compact.

Definition 25. More generally, we call a Siegel set any subset of $GL_n(\mathbb{R})$ of the form $S_{t,\omega} = K \cdot A_t \cdot \omega$, where $\omega$ is a compact neighbourhood of $e$ in $N$.

Remark 26. Let $S$ be a Siegel set. Then for any $g \in K$ or $g = c \cdot I$ ($c > 0$), $g \cdot S = S$, and for any $h \in A \cdot N$, $S \cdot h$ is contained in a Siegel set.
Lemma 27. (11, Lemma 1.3) If \( \omega \) is relatively compact in \( N \), then \( \bigcup_{a \in A_t} a \omega a^{-1} \) is also relatively compact in \( N \).

Proof. In fact, if \( n = (n_{ij}) \in \omega \), then \( (a \omega a^{-1})_{ij} = a_{ii} \cdot n_{ij} \cdot a_{jj}^{-1} \), therefore

\[
\left| (a \cdot n \cdot a^{-1})_{ij} \right| = |a_{ii}/a_{jj}| \cdot |n_{ij}| \leq t^{i-j} \cdot |n_{ij}|
\]

for all \( i < j \).

\[ \square \]

Remark 28. Note that the Haar measure on \( N \) is the image under \( \theta^{-1} \) of the Lebesgue measure on \( \mathbb{R}^{n(n-1)/2} \), so that the modulus of the automorphism \( \text{int}(a) : n \mapsto a \cdot n \cdot a^{-1} \in \text{Aut}(N) \) is

\[
\left| \det_{\mathbb{R}^{n(n-1)/2}}(\text{int}(a)) \right| = \prod_{i < j} \frac{a_{ii}}{a_{jj}}
\]

Theorem 29. (11, Theorem 1.4) For \( t \geq 2/\sqrt{3} \) and \( u \geq 1/2 \), one has \( \text{GL}_n(\mathbb{R}) = \mathcal{G}_{t,u} \cdot \text{GL}_n(\mathbb{Z}) \).

Proof. We first note that

\[
(4.1) \quad N = N_{1/2} \cdot N_{\mathbb{Z}}
\]

, where \( N_{\mathbb{Z}} := N \cap \text{GL}_n(\mathbb{Z}) \). This translates to: given \( u = (u_{ij}) \in N \), one can find \( z = (z_{ij}) \in N_{\mathbb{Z}} \) such that \( |(u \cdot z)_{ij}| \leq 1/2 \). Now, one has

\[
(u \cdot z)_{ij} = z_{ij} + u_{i,i+1} \cdot z_{i+1,j} + \ldots + u_{i,j}
\]

for all \( 1 \leq i < j \leq n \), which allows us to construct \( z_{ij} \) recursively, starting from \( z_{n-1,n} \), which is constructed by rounding \( u_{n-1,n} \).

The essential point is thus the condition imposed on the component in \( A_t \). To treat it, we utilize the minimum principle. Let \( (e_i)_{i=1}^n \) be the standard basis for \( \mathbb{R}^n \), and let \( \Phi \) be the function on \( \text{GL}_n(\mathbb{R}) \) defined by \( \Phi(g) = \| g \cdot e_1 \| \) (Euclidean norm). This is a continuous positive real function, satisfying

\[
\Phi(k \cdot a \cdot n) = \| k \cdot a \cdot n(e_1) \| = \| a(e_1) \| = a_1 = \Phi(a)
\]

for any \( k \in K, a \in A_t, n \in N \), where \( a_1 \) is the first entry of \( a \).

For any \( g \in \text{GL}_n(\mathbb{R}) \), the function \( z \mapsto \Phi(g \cdot z) \ (z \in \text{GL}_n(\mathbb{Z})) \) attains a minimum over \( \text{GL}_n(\mathbb{Z}) \).

In fact, \( g \cdot \text{GL}_n(\mathbb{Z}) \cdot e_1 \subseteq g(\mathbb{Z}^n - \{0\}) \), so that \( g \cdot \text{GL}_n(\mathbb{Z}) \cdot e_1 \) consists of non-zero elements of a lattice in \( \mathbb{R}^n \).

\[ \square \]

Lemma 30. (11, Lemma 1.5) Let \( g \in \text{GL}_n(\mathbb{R}) \), and let \( g = k \cdot a \cdot n \) be its Iwasawa decomposition. Suppose that \( \Phi(g) \leq \Phi(g \cdot \gamma) \) for \( \gamma \in \text{GL}_n(\mathbb{Z}) \). Then \( a_{11} \leq (2/\sqrt{3}) \cdot a_{22} \).

Proof. Let \( u \in N_{\mathbb{Z}} \). Then \( \Phi(g \cdot u) = \Phi(g) \) and \( a_{g \cdot u} = a_g \). Therefore, we may assume, by (4.1), that \( |n_{12}| \leq 1/2 \). Let \( z \in \text{GL}_n(\mathbb{Z}) \) be the element permuting \( e_1 \) and \( e_2 \), and fixing all the other \( e_i \)'s (\( 3 \leq i \leq n \)). Then

\[
g \cdot z(e_1) = g(e_2) = k \cdot a \cdot n(e_2) = k \cdot a \cdot (n_{12} \cdot e_1 + e_2) = k \cdot (a_{22} \cdot e_2 + a_{11} \cdot n_{12} \cdot e_1)
\]
Thus

\[ \Phi(g \cdot z)^2 = \|a_{22} \cdot e_2 + a_{11} \cdot n_{12} \cdot e_1\|^2 = a_{22}^2 + a_{11}^2 n_{12}^2 \leq a_{11}^2/4 + a_{22}^2 \]

As \( \Phi(y) = a_{11} \), we see that

\[ a_{11}^2 \leq a_{11}^2/4 + a_{22}^2 \]

, hence the Lemma.

\[ \square \]

Theorem 29 is now a consequence of the more precise theorem.

**Theorem 31.** ([1], Theorem 1.6) Let \( g \in GL_n(\mathbb{R}) \). The minimum of \( \Phi \) over \( g \cdot GL_n(\mathbb{Z}) \) is attained at a point of \( g \cdot GL_n(\mathbb{Z}) \cap \mathfrak{S}_{2/\sqrt{3},1/2} \).

**Proof.** Write \( \mathfrak{S}_0 \) for \( \mathfrak{S}_{2/\sqrt{3},1/2} \). The proof proceeds by induction on \( n \). For \( n = 1 \), \( GL_n(\mathbb{R}) = \mathbb{R}^* = \mathfrak{S}_0 \), and there is nothing to show.

Let \( x \in GL_n(\mathbb{R}) \). One can find \( y \in x \cdot GL_n(\mathbb{Z}) \) such that \( \Phi(y) \leq \Phi(x \cdot \gamma) \) for all \( \gamma \in GL_n(\mathbb{Z}) \). (by the minimum principle, see above), from which also \( \Phi(y) \leq \Phi(y \cdot \gamma) \) for all \( \gamma \in GL_n(\mathbb{Z}) \). We can write

\[ k_y^{-1} \cdot y = \begin{pmatrix} a_{11} & * \\ 0 & b \end{pmatrix}, \quad b \in GL_{n-1}(\mathbb{R}) \]

By the induction hypothesis, there exists \( z' \in GL_{n-1}(\mathbb{R}) \) such that \( b \cdot z' \in \mathfrak{S}_0^{(n-1)} \).

Let

\[ b \cdot z' = k' \cdot a' \cdot n' \]

be the Iwasawa decomposition of \( k' \cdot a' \cdot n' \). Then

\[ k_y^{-1} \cdot y \cdot z = \begin{pmatrix} a_{11} & * \\ 0 & k' \cdot a' \cdot n' \end{pmatrix} = k'' \cdot a'' \cdot n'', \quad z = \begin{pmatrix} 1 & 0 \\ 0 & z' \end{pmatrix} \]

with

\[ k'' = \begin{pmatrix} 1 & 0 \\ 0 & k' \end{pmatrix} \in K, \quad a'' = \begin{pmatrix} a_{11} & 0 \\ 0 & a' \end{pmatrix} \in A, \quad n'' = \begin{pmatrix} 1 & * \\ 0 & n' \end{pmatrix} \in N \]

By construction, one has \( (a'')_{ii} \leq (2/\sqrt{3}) \cdot (a'')_{i+1,i+1} \) for all \( 2 \leq i < n \). But \( z \) fixes \( e_1 \), hence \( \Phi(y \cdot z) = \Phi(y) \), and hence:

\[ \Phi(y \cdot z) \leq \Phi(y \cdot z \cdot \gamma) \quad \gamma \in GL_n(\mathbb{Z}) \]

Lemma 30 then shows that \( (a'')_{11} \leq (2/\sqrt{3}) \cdot (a'')_{22} \). The consequence is that \( y \cdot z \in K \cdot A_{2/\sqrt{3},1,1} \cap \mathfrak{S}_0 \cdot GL_n(\mathbb{Z}) \), and by (4.1)

\[ x \in y \cdot GL_n(\mathbb{Z}) \subseteq K \cdot A_{2/\sqrt{3},1,1} \cdot GL_n(\mathbb{Z}) = \mathfrak{S}_0 \cdot GL_n(\mathbb{Z}) \]

\[ \square \]

**Corollary 32.** (Hermite, [1], Corollary 1.7) Let \( g \in GL_n(\mathbb{R}) \). Then

\[ \min_{x \in \mathbb{Z}^n \setminus \{0\}} \|g(x)\| \leq (2/\sqrt{3})^{(n-1)/2} \cdot \|\det g\|^{1/n} \]
Proof. By Theorem 31, one can find an element $g' \in g \cdot GL_n(\mathbb{Z}) \cap \mathfrak{S}_{2/\sqrt{3},1/2}$ in which the minimum of $\Phi$ over $g \cdot GL_n(\mathbb{Z})$ is attained. As $\det(GL_n(\mathbb{Z})) = \{ \pm 1 \}$, we have $|\det g| = |\det g'|$. On the other hand

$$\min_{x \in \mathbb{Z}^n - \{0\}} \|g(x)\| \leq \min_{\gamma \in GL_n(\mathbb{Z})} \|g\gamma(e_1)\| = \|g'(e_1)\| = a'_{11}$$

where $a' \in A$ is the Cartan component in the Iwasawa decomposition of $g'$. Since $a' \in A_{2/\sqrt{3}}$, we get

$$(a'_{11})^n \leq (2/\sqrt{3})^{n(n-1)/2} \cdot a'_{11} \cdot a'_{22} \cdot \cdots \cdot a'_{nn} = (2/\sqrt{3})^{n(n-1)/2} \cdot |\det g|$$

\[\square\]

Remark 33. We were using Euclidean norm ($L^2$), but as all norms are equivalent on $\mathbb{R}^n$, it shows that for any norm, there exists a constant $C$ replacing the $(2/\sqrt{3})^{n(n-1)/2}$ such that the claim holds.

Definition 34. Let $\Delta$ be the function on $\mathfrak{R}$ associating to every lattice $L$, the Euclidean volume of its fundamental parallelotope. If $L = g(L_0)$, with $L_0 = \mathbb{Z}^n$, then $\Delta(L) = |\det g|$.

Corollary 35. (Mahler's Criterion, [1], Corollary 1.9) Let $M \subseteq \mathfrak{R}$. TFAE:

(a) $M$ is relatively compact.

(b) $\Delta$ is bounded over $M$ and there exists a neighbourhood $U$ of the origin in $\mathbb{R}^n$ such that $L \cap U = \{0\}$ whenever $L \in M$.

Proof. Let $\mathfrak{S}$ be a Siegel set, sent onto $\mathfrak{R}$ by the map $g \mapsto g(L_0)$ (exists, by Theorem 29). Then (a) is equivalent to the existence of $M' \subseteq \mathfrak{S}$, relatively compact such that $M'(L_0) = M$. (Why? If it exists, $M$ is its continuous image, hence rel. compact. Conversely, if $M$ is rel. compact, this map is proper).

On the other hand, $M'$ is relatively compact iff the components $\{a_x\}_{x \in M'}$ form a relatively compact set in $A$. (because $N_{ii}$ is compact). Therefore this happens iff there are $\alpha, \beta > 0$ such that for all $g \in M'$: $\alpha \leq (a_g)_{ii} \leq \beta$. It now remains to show that this is equivalent to:

(Why? a nbd of the origin is some $B(0,c)$ such that for $L \in M$, and any $x \in L$, $|x| \geq c$, pulling back through the map we get it)

(*) $|\det g|$ is bounded from above on $M'$; There exists $c > 0$ such that $\|g(x)\| \geq c$ for all $x \in \mathbb{Z}^n - \{0\}$, $g \in M'$.

$\Rightarrow$ One has $|\det g| = \prod |(a_g)_{ii}|$, hence it is bounded. Let $x \in \mathbb{Z}^n - \{0\}$. One can write $x = \sum_{i=1}^k m_i e_i$ with $m_i \in \mathbb{Z}$, $m_k \neq 0$.

Then $\|g(x)\| = \|a_g \cdot n_g(x)\|$ and the $k$th coordinate of $a_g \cdot n_g(x)$ is $(a_g)_{k,k} m_k$. Therefore, $\|g(x)\| \geq \alpha$.

$\Leftarrow$ One has $\|g(e_1)\| = (a_g)_{11} \geq c$. As the $a_g$ ($g \in M'$) are a subset of the set $A_2$, it shows the existence of a constant $c$ such that $(a_g)_{ii} \geq \alpha$ for all $i$. Since the product is bounded, the result follows. $\square$
4.3.2. Algebraic groups. With Mahler’s criterion in our hand, we still need to understand when the embedding $G(\mathbb{R})/G_Z \hookrightarrow \mathbb{R}$ is closed. (to deduce compactness of $G(\mathbb{R})/G_Z$).

For that we will need some statements about algebraic (reductive, semisimple) groups.

Fiest, we will show that under nice enough hypotheses, our group $G$ has a nice representation.

Recall that a torus is a connected commutative group formed of semisimple elements, which is the same as being connected and diagonalizable, i.e. isomorphic over the algebraic closure to $G_m^n$ for some $n \in \mathbb{Z}$. Also, the radical is the maximal connected solvable normal subgroup. A group with a trivial radical is said to be semisimple.

**Definition 36.** An algebraic group $G$ is reductive if its identity component $G^0$ is equal to an almost direct product (i.e. $S \times H \to G^0$ is an isogeny) $S \cdot H$ of a central torus $S$ by a semisimple group $H$.

**Remark 37.** In characteristic 0, this is equivalent to complete reducibility of the representations.

**Proposition 38.** ([1], Proposition 7.6) Let $G$ be a reductive group acting on a vector space $W$, and $X$ an irreducible closed subvariety of $W$, stable under $G$. Let $\bar{k}$ be an algebraically closed field of characteristic 0. Let $I := \bar{k}[X]^G$ be the ring of $G$-invariant regular functions on $X$.

(i) There exists a projection $\pi : \bar{k}[X] \to I$, which is $I$-linear and keeps every $G$-invariant subspace stable.

(ii) $I$ separates the $G$-invariant closed algebraic sets of $X$.

(iii) $I$ is of finite type over $\bar{k}$.

**Proof.** (i) Let $N$ be the sum of minimal subspaces of $\bar{k}[X]$ over which $G$ does not act trivially. Since the action of $G$ on $\bar{k}[X]$ is completely reducible, we have $\bar{k}[X] = I \oplus N$. We define the projection $\pi : \bar{k}[X] \to I$ as the projection with respect to this decomposition. We have to show that it is $I$-linear.

First note that $I \cdot N \subseteq N$, hence $I \cdot N \cap I = \{0\}$.

If $\varphi \in I$ and $\psi \in \bar{k}[X]$, then write $\psi = \psi^I + \psi_N$, so $\varphi \cdot \psi = \varphi \cdot \psi^I + \varphi \cdot \psi_N$, hence

$$(\varphi \cdot \psi)^I = \varphi \cdot \psi^I + (\varphi \cdot \psi_N)^I = \varphi \cdot \psi^I$$

Finally, let $E$ be a $G$-invariant subspace of $\bar{k}[X]$. Then $E = (E \cap I) \oplus (E \cap N)$, hence $E^I \subseteq E$.

(ii) Let $A, B$ be two $G$-invariant closed algebraic sets of $X$, with ideals $I(A), I(B)$. Assume $A \cap B = \emptyset$, hence by Nullstellensatz, $I(A) + I(B) = \bar{k}[X]$. Then there exist $\alpha \in I(A), \beta \in I(B)$ such that $\alpha + \beta = 1$, hence $\alpha^2 + \beta^2 = 1$. Now, the fact that $A$ and $B$ are $G$-stable, means that $I(A)$ and $I(B)$ are $G$-stable. Therefore, $\alpha^2 \in I(A)$, $\beta^2 \in I(B)$ and $\alpha^2$ vanishes on $A$, and is equal to 1 on $B$.

(iii) We identify $\bar{k}[X]$ as a quotient of $\bar{k}[W]$ by the ideal defining $X$ and the projection $\pi : \bar{k}[W] \to \bar{k}[X]$ commutes with $G$. By complete reducibility we get

$$\pi(\bar{k}[W]^G) = I$$
It is therefore enough to prove (iii) when $W = X$. In this case, the functions of $I$ which vanish at the origin generate an ideal, which has a finite generating set $f_1, \ldots, f_s$. As the homogeneous components of an invariant polynomial are also invariant, we may assume that the $f_i$ are homogeneous, and it is enough to show that every homogeneous $f \in I$ of degree $> 0$ is a polynomial in the $f_i$. One may write $f = \sum_{i=1}^s a_i f_i$ with $a_i \in \overline{k} [W]$ homogeneous of degree $\deg(a_i) = \deg(f) - \deg(f_i)$. Applying $\Psi$, we see that $f = \sum a_i^s f_i$, so we get $a_i \in I$, and we can now continue by recursion on the degree.

**Proposition 39.** ([1], Proposition 7.7) Let $G$ be a connected algebraic group, $H$ a reductive subgroup of $G$, and $k$ a field of definition for both $G$ and $H$. Then there exist: a finite dimensional vector space $W$ defined over $k$, a $k$-rational right representation of $G$ on $W$, and a point $w \in W(k)$, such that the orbit of $w$, $w \cdot G$, is closed, and its isotropy group is $H$.

**Proof.** The reductive group $H$ acts by left multiplication on the closed subvariety $G \subseteq GL_V$. The algebra $I := \overline{k}[G]^H$ of functions, which are constant on right cosets $Hx$ of $G$ by $H$, admit a finite generating set $w_1, \ldots, w_s$, that one can take to lie in $k[G]$, by Proposition 38 (iii). These belong to subspaces $W_i$ of $\overline{k}[G]$, that are defined over $k$, stable by $G$, and finite dimensional.

Consider the vector space $W = \bigoplus_{i=1}^s W_i$, equipped with the right representation

$$(v_1, \ldots, v_s) \mapsto (v_1 \cdot g, \ldots, v_s \cdot g)$$

and the point $w \in W(k)$ given by $w = (w_1, \ldots, w_s)$. We have to prove that $G_w = H$. Clearly $H \subseteq G_w$, since $w_i \in I$ and so $w_i \cdot g = w_i$ for all $i$, and $w \cdot g = w$ for all $g \in H$. Conversely, if $g \in G_w$, then $w_i \cdot g = w_i$ for all $i$, and as this is a generating set for $I$, we see that $f \cdot g = f$ for all $f \in I$, hence $f(g) = f(e)$. But by Proposition 38 (ii), $I$ separates the $H$-invariant algebraic closed sets of $G$, in particular the right cosets of $G$ by $H$. This implies that $g \in H$.

It remains to show that the orbit $w \cdot G$ is closed. The morphism $\varphi : G \to W$ defined by $g \mapsto w \cdot g$ correspond to the comorphism

$$\varphi^2 : A := \overline{k}(\overline{w \cdot G}) \to \overline{k}[G]$$

In fact, $\varphi^2(A) \subseteq I$, because the inclusion $H \subseteq G_w$ implies

$$[\varphi^2(f) : h](g) = \varphi^2(f)(h \cdot g) = f(w \cdot h \cdot g) = f(w \cdot g) = \varphi^2(f)(g)$$

The map $\varphi^2$ is injective, because if $\varphi^2 f = 0$, then $f$ is zero on $w \cdot G$, hence on $\overline{w \cdot G}$. Finally, its image is $I$; to see this, it is enough to show that every generator $w_i$ of $I$ belongs to $\varphi^2(A)$. If $\{z_1, \ldots, z_n\}$ is a basis of $W_i$ and $\{a_1, \ldots, a_n\}$ its dual basis, they define functions on $W$:

$$u_j : (v_1, \ldots, v_s) \mapsto a_j(v_i)$$

such that $(\varphi^2 u_j)(g) = a_j(w_i \cdot g)$. It follows that

$$w_i(g) = (w_i \cdot g)(e) = \sum_j a_j(w_i \cdot g)z_j(e) = \sum_j z_j(e)(\varphi^2 u_j)(g)$$

hence $\varphi^2$ is an isomorphism from $A$ onto $I$.

In order to prove $w \cdot G = \overline{w \cdot G}$, it is enough to prove for every $x \in \overline{w \cdot G}$ the existence of $y \in w \cdot G$ which is a zero of $m_x$, the maximal ideal at $x$. By the
above isomorphism, it is enough to find \( g \in G \) which is a zero of \( \varphi^2(m_x) \). For that, it suffices to know that \( \varphi^2(m_k[G]) \) is a proper ideal of \( k[G] \) for every proper ideal \( m_k \subseteq A \). If it weren't the case, there would have been \( m_1, \ldots, m_t \in m \) and \( f_1, \ldots, f_t \in k[G] \) such that \( \sum \varphi^2(m_i)f_i = 1 \), which due to \( \varphi^2(A) \subseteq I \) and 7.6(1) means that
\[
\sum (\varphi^2(m_i) \cdot f_i^2) = 1
\]
But \( \varphi^2 \) is invertible, so that \( \sum m_i f_i^2 = 1 \) for some \( f_i' \in A \), contradicting the fact that \( m \) is a proper ideal of \( A \).

**Proposition 40.** ([1], Proposition 7.8) Let \( G \) be an algebraic group, \( H \) a closed subgroup, \( k \) a field of definition for \( G, H \). Then there exists a finite dimensional vector space \( W \), defined over \( k \), and a morphism \( G \to GL_W \), defined over \( k \), and a point \( w \in W(k) \) such that \( H \) is the set of elements of \( G \) that stabilize the line \([w] \).

**Proof.** Let \( J \subseteq \bar{k}[G] \) be the ideal of functions vanishing on \( H \). It is defined over \( k \), and has a finite set of generators. Thus, there exists a finite dimensional subspace \( V \subseteq \bar{k}[G] \), defined over \( k \), stable by left translation, such that \( V \cap J \) generates \( J \). One can then take \( W = \bigwedge^d V \), where \( d = \dim(V \cap J) \), \([w]\) the line corresponding to \( V \cap J \), and \( \rho \) the induced representation on \( W \) by the given representation of \( G \) on \( V \).

**Corollary 41.** ([1], Corollary 7.9) If \( X(H)_k = \{1\} \), then there exists a morphism \( G \to GL_W \), defined over \( k \), and a point \( w \in W(k) \), such that its isotropy group is \( H \).

Finally, this “nice” representation gives us the compactness property that we were looking for.

**Theorem 42.** ([1], Theorem 8.1) Let \( G \subseteq GL_n \) be algebraic group over \( \mathbb{Q} \), which is either reductive or such that \( X(G)_{\mathbb{Q}} = \{1\} \). Then \( G(\mathbb{R})/G(\mathbb{Z}) \) is closed in \( \mathfrak{X} \).

**Proof.** By definition of the quotient topology, it suffices to prove that \( G(\mathbb{R}) \cdot GL_n(\mathbb{Z}) \) is closed in \( GL_n(\mathbb{R}) \). The assumption here shows, by Proposition 39 and Corollary 41 that \( G \) has the following property:

(P) There exists an algebraic right representation \( \pi : GL_n \to GL_V \), defined over \( \mathbb{Q} \), and an element \( v \in V \) such that the isotropy group of \( v \) is \( G \).

By Corollary 18 (1), there exists a lattice \( L \) of \( V \), containing \( v \) and stable by \( GL_n(\mathbb{Z}) \), hence \( v \cdot GL_n(\mathbb{Z}) \) is closed in \( V(\mathbb{R}) \).

Now \( G(\mathbb{R}) \cdot GL_n(\mathbb{Z}) \) is the inverse image of \( v \cdot GL_n(\mathbb{Z}) \) in \( GL_n(\mathbb{R}) \) under the map \( g \mapsto v \cdot g \). Therefore, \( G(\mathbb{R}) \cdot GL_n(\mathbb{Z}) \) is closed.

Combining it with what we know on \( \mathfrak{X} \), we get:

**Proposition 43.** ([1], Proposition 8.2) Let \( G \subseteq GL_n \) be a reductive algebraic group over \( \mathbb{Q} \), \( \Gamma \) an arithmetic subgroup of \( G \), and \( M \subseteq G(\mathbb{R}) \) a subset. TFAE:

(i) \( M \) is relatively compact modulo \( \Gamma \).

(ii) \( M \) is relatively compact modulo any arithmetic subgroup of \( G \).

(iii) \( |\det g| \) has an upper bound over \( M \) and there exists a constant \( c > 0 \) such that \( |g(x)| \geq c \) where \( g \in M \) and \( x \in \mathbb{Z}^n - \{0\} \).
(iv) \(|\det g|\) has an upper bound over \(M\); If \((v_j)_{j \in \mathbb{N}}\) and \((g_j)_{j \in \mathbb{N}}\) are two sequences of elements of a lattice \(L \subseteq \mathbb{Q}^n\) and of \(M\), respectively, such that \(g_j v_j \to 0\) when \(j \to \infty\), then \(v_j = 0\) for \(j\) large enough.

**Proof.** The equivalence of (i) and (ii) follows from the fact that any two arithmetic subgroups are commensurable.

In the case \(\Gamma = G_{\mathbb{Z}}\), (i) is equivalent to saying that the image of \(M\) in \(\mathfrak{X}\) is relatively compact. Then its equivalence to (iii) follows from Mahler’s criterion (Corollary 35). Finally (iv) is equivalent to (iii) after changing the lattice. \(\square\)

**Proposition 44.** ([1], Proposition 8.5) Let \(G, G'\) be \(\mathbb{Q}\)-groups, \(\rho : G \to G'\) a surjective \(\mathbb{Q}\)-morphism. Suppose \(G\) is reductive, \(X(G^o)_{\mathbb{Q}} = \{1\}\) and \(\ker \rho\) is commutative. Let \(\Gamma, \Gamma'\) be arithmetic subgroups of \(G, G'\) such that \(\rho(\Gamma) \subseteq \Gamma'\), and let \(D = \rho^{-1}(\Gamma') \cap G(\mathbb{R})\). Then \(D/\Gamma\) is compact, and the map \(G(\mathbb{R})/\Gamma \to G'(\mathbb{R})/\Gamma'\) induced by \(\rho\) is proper.

**Proof.** The map \(G(\mathbb{R})/\Gamma \to G'(\mathbb{R})/\Gamma'\) factorizes through \(G(\mathbb{R})/\Gamma \to G(\mathbb{R})/D \to G'(\mathbb{R})/\Gamma'\). The first map is a fibration with fibre \(D/\Gamma\), and is proper iff \(D/\Gamma\) is compact. The second is injective onto a closed set (Wait, why is it closed? by Theorem 42), therefore proper. Thus, it is enough to see that \(D/\Gamma\) is compact.

Since any two arithmetic groups of a \(\mathbb{Q}\)-group are commensurable, we see that it is enough to prove for a single pair. Indeed, if it holds for \(\Gamma_1, \Gamma_1\), then \(\rho^{-1}(\Gamma_1') \cap G(\mathbb{R})/\Gamma_1\) is compact. We have

\[
\rho^{-1}(\Gamma') \cap G(\mathbb{R})/\Gamma_1 = \bigcup_{x \in \rho^{-1}(\Gamma') \cap G(\mathbb{R})/\rho^{-1}(\Gamma_1') \cap G(\mathbb{R})} x \cdot \rho^{-1}(\Gamma_1') \cap G(\mathbb{R})/\Gamma_1
\]

a finite union of compacts is also compact, and \(\rho^{-1}(\Gamma') \cap G(\mathbb{R})/\Gamma_1 \to \rho^{-1}(\Gamma') \cap G(\mathbb{R})/\Gamma\) is a surjection with finite fibers \((\Gamma/\Gamma_1)\). In particular, it is continuous, and so the image is also compact.

Moreover, since \(|G : G'| < \infty\), we may reduce to the case of \(G\) connected. Then \(\ker \rho \subseteq C\), the center of \(G\), and the projection \(G \to G/C\) factors through \(\rho\).

Let \(\Gamma^{ad}\) be an arithmetic subgroup of \(G^{ad} = G/C\) containing the image of \(\Gamma'\). (by Corollary 18 (3)) We have

\[
G(\mathbb{R})/\Gamma \xrightarrow{\alpha} G'(\mathbb{R})/\Gamma' \xrightarrow{\beta} G^{ad}(\mathbb{R})/\Gamma^{ad}
\]

In order to show that \(\alpha\) is proper, it’s enough to show that \(\beta \circ \alpha\) is proper. Therefore, it suffices to consider the case where \(G' = G^{ad} = G/C\).

Let \(i : G \hookrightarrow GL_n\) be an embedding defined over \(\mathbb{Q}\). \(C\) defines a commutative family of semisimple (why s.s.?) endomorphisms of \(\mathbb{C}^n\). Then \(\mathbb{C}^n\) is a semisimple \(C\)-module (common eigenvectors), and \(A\), the commutant of \(C\) in \(M_n(\mathbb{C})\), is a semisimple (by structure theorem) \(\mathbb{C}\)-algebra, containing \(G(\mathbb{C})\).

Let \(\beta : G \to GL_A\) be the adjoint map, sending \(g\) to the inner automorphism \(\beta(g) : x \mapsto gxg^{-1}\). Since everything is defined over \(\mathbb{Q}\), \(\beta : G \to GL_A\) is a \(\mathbb{Q}\)-morphism with image \(G' = G/C\). The intersection \(L = M_n(\mathbb{Z}) \cap A\) is an order in \(A(\mathbb{Q})\), and in particular a lattice. We identify \(GL_A\) with \(GL_m\), using a basis of \(L\), and we take \(\Gamma, \Gamma'\) to be the groups \(G_0\) and \(G_0^\perp\), respectively. Then \(\beta(\Gamma) \subseteq \Gamma'\). (Why?) Recall \(\Gamma' = G' \cap GL_m(\mathbb{Z}) = G'_1\). If \(g \in \Gamma = G_{\mathbb{Z}} = GL_n(\mathbb{Z})\), and \(x \in M_n(\mathbb{Z}) \cap A\), then \(\beta(g)(x) = gxg^{-1} \in M_n(\mathbb{Z})\). Hence, \(\beta(g) \in \Gamma'\)
Moreover, \( D = \{ g \in G(\mathbb{R}) \mid gLg^{-1} = L \} \). (Indeed, \( D = \beta^{-1}(\Gamma') \cap G(\mathbb{R}) = \{ g \in G(\mathbb{R}) \mid \beta(g) \in G'_L \} \))

To prove that \( D/\Gamma \) is compact, we apply Mahler’s criterion to \( G \) acting on \( A \) by left translations. Let \( \lambda(g) \) be the left translation on \( A \) by \( g \in G \) \( (x \mapsto gx) \). Let \( \mathfrak{R} \) be the space of lattices in \( A(\mathbb{R}) \). Since \( A \) contains the identity, the stabilizer of \( L \) in \( G(\mathbb{R}) \) is \( \Gamma \). \( (g \cdot (M_n(\mathbb{Z}) \cap A) = M_n(\mathbb{Z}) \cap A \iff g \in GL_n(\mathbb{Z}) \).

The group \( X(G|_\mathfrak{R}) \) is trivial by assumption, hence \( \det \lambda(g) = 1 \). By Theorem 42 \( G(\mathbb{R})/\Gamma \) is closed in \( \mathfrak{R} \), and by Proposition 43, it’s enough to check that if \( (x_j) \) and \( (y_j) \) are sequences of elements of \( L \) and \( D \), respectively, such that \( g_jx_j \to 0 \), then \( x_j = 0 \) for all \( j \) large enough.

For that we can replace \( L \) by an arbitrary sublattice.

(Why? Because of the statement of Proposition 43)

\( A \) is the direct sum of its two-sided minimal ideals defined over \( \mathbb{Q} \), and the direct sum of the \( L_i = L \cap A_i \) is a lattice in \( A(\mathbb{Q}) \) contained in \( L \). It follows that one may restrict to the case where the \( \{ x_j \} \) is a sequence of elements of \( L_i \), for a fixed \( i \).

(Why? assume first that \( L \) is the sum, later use previous reduction step. Suppose that \( x_j = \sum x_{ij} \), with \( x_{ij} \in L_i \). Then \( \sum g_jx_{ij} \to 0 \). However, \( L_i \subseteq \mathbb{R} \) is a two-sided ideal, so that \( g_jL_i \subseteq g_jA_i = A_i \), and we see that \( g_jx_{ij} \in L_i \). Since this is a direct sum, we get for each \( i \), \( g_jx_{ij} \to 0 \), with \( \{ x_{ij} \} \in L_i \) and \( g_j \in D \). Note also that by the above \( g_jL_i, g_j^{-1} \subseteq L_i \) and also \( g_j^{-1}L_i g_j \subseteq L_i \), so that \( D \subseteq \{ g \in G(\mathbb{R}) \mid gL_i g^{-1} = L_i \} \)

Put on \( A(\mathbb{R}) \) the norm \( \| a \| = \operatorname{tr}(a' \cdot a) \). Then \( \| a \cdot b \| \leq \| a \| \cdot \| b \| \) and \( \| a \cdot b \| = \| b \cdot a \| \).

It’s then enough to show:

(*) There exists a constant \( c > 0 \) such that \( \| d \cdot x \| \leq c \) \( (d \in D, x \in L_i) \) implies \( x = 0 \).

By Hermite’s Theorem (Corollary 32), there exists a constant \( c_1 > 0 \) such that

\[
\min_{x \in L_i - \{0\}} \| g \cdot x \| \leq c_1 \cdot |\det g|^{1/n}, \quad (g \in GL(A_i(\mathbb{R})), n = \dim A_i)
\]

Since \( \det g = 1 \) for \( g \in G \), it shows that for \( d \in D \), one can find an element \( z_d \in L_i - \{0\} \) such that \( \| z_d \cdot d^{-1} \| \leq c_1 \). (Why? For \( d \in D \), we have \( dL_i = L_id \), so by the above we can find some \( y_d \in L_i - \{0\} \) for which \( \| d^{-1} \cdot y_d \| \leq c_1 \). But \( d^{-1}y_d = z_d \cdot d^{-1} \) for some \( z_d \in L_i - \{0\} \)).

Choose a basis \( \{ y_i \} \) of \( L_i \), and let \( c_2 = \min_{x \in L_i - \{0\}} \| x \|, c_3 = \max \| y_i \| \). We would like to show that (*) holds for all \( c \) satisfying \( 0 < c < c_2/c_1c_3 \).

Let \( d \in D \) and \( x \in L_i \) be such that \( \| d \cdot x \| \leq c \). The lattice \( L_i \) is an order in \( A_i(\mathbb{Q}) \).

Thus, \( xy_d z_d \in L_i \), from which by definition of \( D \):

\[
dxy_d z_d d^{-1} \in L_i
\]

Let \( z_d \) be chosen of highest norm. We see that

\[
\| dxy_d z_d d^{-1} \| \leq \| d \cdot x \| \| y_i \| \| z_d \cdot d^{-1} \| \leq c \cdot c_3 \cdot c_1 < c_2
\]

Therefore, \( dxy_d z_d d^{-1} = 0 \), so that \( dxy_d z_d = 0 \).

As this holds for any element \( y_j \) in the basis of \( L_i \), we get \( x \cdot A_i \cdot z_d \cdot A_i = 0 \). But \( z_d \neq 0 \) is rational over \( \mathbb{Q} \), and \( A_i \) is a minimal two-sided ideal over \( \mathbb{Q} \), hence \( A_i \cdot z_d \cdot A_i = A_i \), hence \( x \cdot A_i = 0 \), so that \( x = 0 \). \( \square \)
In order to generalize results on reductive groups to arbitrary algebraic groups, one needs the following theorem:

**Theorem 45.** ([2], Proposition 5.1) Let $k$ be a field of characteristic 0, and $G$ a $k$-group. Then $G$ is a semidirect product of a reductive $k$-group $H$ and a connected normal unipotent $k$-group $N$. Every $k$-subgroup of $G$ is conjugate by an element of $N(k)$ to a subgroup of $H$.

**Remark 46.** The subgroup $N$ is uniquely determined by the theorem, and is the unipotent radical of $G$, i.e. the maximal connected normal subgroup of $G$ consisting of unipotent elements, which is denoted by $R_u(G)$. It is a nilpotent group. Also, since $N$ is connected, $G$ is connected iff $H$ is. In particular $G^o = H^o \cdot N$.

**Theorem 47.** ([1], Theorem 8.9) Let $\rho : G \to G'$ be a $\mathbb{Q}$-isogeny of $\mathbb{Q}$-groups. If $\Gamma \subseteq G(\mathbb{Q})$ is arithmetic, then so is $\rho(\Gamma) \subseteq G'(\mathbb{Q})$.

**Proof.** It is clear (to Borel - not to me) that it suffices to consider the case where $G$ is connected.

(Why? Assume it is true for connected $G$. Consider $\Gamma^0 := \Gamma \cap G^o(\mathbb{Q})$. $\Gamma$ is arithmetic, hence commensurable with $G(\mathbb{Q}) \cap GL_n(\mathbb{Z})$, so that $[G(\mathbb{Q}) \cap GL_n(\mathbb{Z}) : \Gamma \cap GL_n(\mathbb{Z})] < \infty$ and $[\Gamma : \Gamma \cap GL_n(\mathbb{Z})] < \infty$. Now, the map

$$G^0(\mathbb{Q}) \cap GL_n(\mathbb{Z})/\Gamma^0 \cap GL_n(\mathbb{Z}) \to G(\mathbb{Q}) \cap GL_n(\mathbb{Z})/\Gamma \cap GL_n(\mathbb{Z})$$

is injective, so that

$$[G^0(\mathbb{Q}) \cap GL_n(\mathbb{Z}) : \Gamma^0 \cap GL_n(\mathbb{Z})] \leq [G(\mathbb{Q}) \cap GL_n(\mathbb{Z}) : \Gamma \cap GL_n(\mathbb{Z})] < \infty$$

and the map $\Gamma^0/\Gamma^0 \cap GL_n(\mathbb{Z}) \to \Gamma/\Gamma \cap GL_n(\mathbb{Z})$ is also injective, so that $[\Gamma^0 : \Gamma^0 \cap GL_n(\mathbb{Z})] < \infty$. Thus $\Gamma^0$ is arithmetic in $G^o$.

By the connected case, $\rho(\Gamma^o)$ is arithmetic in $\rho(G^o)$, i.e. commensurable with $\rho(G^o)(\mathbb{Q}) \cap GL_n(\mathbb{Z}) = \rho(G^0(\mathbb{Q})) \cap GL_n(\mathbb{Z})$, so that $[\rho(\Gamma^o) : \rho(\Gamma^0) \cap GL_n(\mathbb{Z})] < \infty$ and $[\rho(\Gamma^0) : \rho(\Gamma^o) \cap GL_n(\mathbb{Z})] < \infty$.

Now, we have

$$[\rho(\Gamma) \cap GL_n(\mathbb{Z}) : \rho(\Gamma^o) \cap GL_n(\mathbb{Z})] \leq [\rho(\Gamma) : \rho(\Gamma^o)] \leq [G : G^o] < \infty$$

so that

$$[\rho(G)(\mathbb{Q}) \cap GL_n(\mathbb{Z}) : \rho(\Gamma) \cap GL_n(\mathbb{Z})] \leq [G : G^o] \cdot [\rho(G)(\mathbb{Q}) \cap GL_n(\mathbb{Z}) : \rho(\Gamma^o) \cap GL_n(\mathbb{Z})] < \infty$$

Also, $[\rho(\Gamma) : \rho(\Gamma^o)] \leq [\rho(G) : \rho(G^o)] < \infty$ going the second finite index.)

By Corollary 18 (3), one can find an arithmetic subgroup $\Gamma'$ of $G'$ such that $\rho(\Gamma) \subseteq \Gamma'$. We will show that this inclusion is of finite index.

Suppose first that $G$ is reductive and $X(G)_{\mathbb{Q}} = \{1\}$. By Proposition 448.5, the quotient $(\rho^{-1}(\Gamma') \cap G(\mathbb{R}))/\Gamma$ is compact. Since $\ker \rho$ is finite, the group $\rho^{-1}(\Gamma')$ is discrete, and so the above quotient is finite, and so $\rho(\Gamma)$ is of finite index in $H := \rho(\rho^{-1}(\Gamma') \cap G(\mathbb{R}))$

But, we also know that $\rho(G(\mathbb{R}))$ is of finite index in $G'(\mathbb{R})$ (It is the connected component of the identity), so that $H$ is of finite index in $\Gamma'$.
Now, assume only that $G$ is reductive (no assumption on characters). Let

$$G_1 := \left( \bigcap_{\chi \in X(G)_Q} \ker \chi \right)^o, \quad G_1' := \left( \bigcap_{\chi \in X(G')_Q} \ker \chi \right)^o$$

For any character $\chi \in X(G)_Q$, one has $\chi(G_2) \in \{ \pm 1 \}$. (Why ??? Complete later)

Hence $G_1 \cap \Gamma$ is of finite index in $\Gamma$, and similarly, $G_1' \cap \Gamma'$ is of finite index in $\Gamma'$.

Also, it can be shown that $\rho(G_1) = G_1'$ and that $X(G_1)_Q = \{ 1 \}$. (Why? $G_1, G_1'$ are the anisotropic tori, this decomposition is unique and preserved under $\mathbb{Q}$-morphisms - this is Borel 10.6 (ii) - maybe insert that part later!!!). Therefore, we are done.

For the general case, one has a decomposition $G = HN$, with $H$ reductive, $N$ a normal unipotent, by 45, and one can assume $\Gamma = \Gamma_1 \Gamma_2$ is the semidirect product of an arithmetic group $\Gamma_1 \subseteq H$ and an arithmetic group $\Gamma_2 \subseteq N$ (by Corollary 18 (4)). The group $G'$ is then the semidirect product of $H' = \rho(H)$ and $N' = \rho(N)$.

Now, $\rho(\Gamma_1)$ is an arithmetic subgroup of $H'$, by what we have already shown. On the other hand, $\ker \rho \cap N = \{ 1 \}$, since $\ker \rho$ is finite (there are no finite order unipotent elements in characteristic zero), so that $\rho$ induces an isomorphism $N \to N'$, and by Corollary 18 (2), $\rho(\Gamma_2)$ is an arithmetic subgroup of $N'$. Therefore $\rho(\Gamma) = \rho(\Gamma_1)\rho(\Gamma_2)$ is an arithmetic subgroup of $G'$, by Corollary 18 (4).

**Corollary 48.** ([1], Corollary 8.10) Suppose $G$ is an almost direct product of normal $\mathbb{Q}$-subgroups $G_i$ ($1 \leq i \leq m$). Then $\Gamma$ is commensurable with the group $\Gamma'$ generated by the intersections $\Gamma_i = \Gamma \cap G_i$, and the $\Gamma_i$ are arithmetic in $G_i$.

**Proof.** It’s enough to consider the case $G \subseteq GL_n$ and $\Gamma = G_\mathbb{Z}$, hence the $\Gamma_i$ are arithmetic. The map $\nu : \prod G_i \to G$ is a $\mathbb{Q}$-isogeny, hence $\nu(\prod \Gamma_i)$ is arithmetic, by Theorem 47, hence commensurable with $\Gamma$. □

We can now extend it.

**Proposition 49.** ([1], Remark 8.11) Let $\rho : G \to G'$ be a surjective homomorphism of algebraic groups over $\mathbb{Q}$. If $\Gamma \subseteq G(\mathbb{Q})$ is arithmetic, then so is $\rho(\Gamma) \subseteq G'(\mathbb{Q})$.

**Proof.** Borel then remarks that this can be extended to any surjective morphism, and shows the proof for the reductive connected case.

Let $N = (\ker \rho)^o$. It is a normal $\mathbb{Q}$-subgroup. Since $G$ is reductive, there exists a connected normal $\mathbb{Q}$-subgroup $N'$ of $G$ such that $G$ is the almost direct product of $N$ and $N'$ (Why is that? Follows from writing semisimple as almost direct product - every connected normal subgroup is a product of the almost simple components). By Corollary 48, $\Gamma$ is commensurable with $(N \cap \Gamma)(N' \cap \Gamma)$ and $N' \cap \Gamma$ is arithmetic in $N'$. The group $\rho(\Gamma)$ is commensurable with $\rho(N' \cap \Gamma)$. As the restriction of $\rho$ to $N'$ is an isogeny of $N'$ on $G'$, $\rho(N' \cap \Gamma)$ is arithmetic by Theorem 47, hence $\rho(\Gamma)$ is arithmetic.

The reduction to this case is the same as in Theorem 47. □
Theorem 50. ([5], Theorem 3.3) Let G be a reductive group over $\mathbb{Q}$, and let $\Gamma$ be an arithmetic subgroup of $G(\mathbb{Q})$.
(a) The space $\Gamma \backslash G(\mathbb{R})$ has finite volume iff $X(G^0)_{\mathbb{Q}} = \{1\}$. (In particular, if $G$ is s.s.)
(b) The space $\Gamma \backslash G(\mathbb{R})$ is compact iff $X(G^0)_{\mathbb{Q}} = \{1\}$ and $G(\mathbb{Q})$ contains no unipotent elements (other than 1).

Let’s see how one proves that... (intuitive reasoning for (b) - unipotent elements correspond to cusps)

We need first to quote the following Lemma:

Lemma 51. ([1], Lemma 8.3) Let $G$ be an algebraic $\mathbb{Q}$-group, $\Gamma$ an arithmetic subgroup, $\pi : G \to GL_V$ a $\mathbb{Q}$-representation. If $G(\mathbb{R})/\Gamma$ is compact then $\pi(G(\mathbb{R})) \cdot v$ is closed in $V(\mathbb{R})$ for any $v \in V(\mathbb{Q})$.

Proof. By hypothesis, $G(\mathbb{R}) = C \cdot \Gamma$, where $C$ is compact, hence $\pi(G(\mathbb{R})) \cdot v = \pi(C) \cdot (\pi(\Gamma) \cdot v)$. It’s therefore enough to show that $\pi(\Gamma) \cdot v$ is closed. But that follows from Corollary 18 (1) (Really? Why?? anyway that should follow from the topological preliminaries). \qed

We have the following: (which is (b))

Theorem 52. ([1], Theorem 8.4) Let $G$ be a reductive group over $\mathbb{Q}$, and let $\Gamma$ be an arithmetic subgroup of $G(\mathbb{Q})$. TFAE:
(i) $G(\mathbb{R})/\Gamma$ is compact.
(ii) $X(G^0)_{\mathbb{Q}} = \{1\}$, and every element of $G_0$ is semisimple.

Proof. First, we reduce to the case of $G$ connected. The group $G^0(\mathbb{R})$ is open and closed, normal, and of finite index in $G(\mathbb{R})$, hence $G^0(\mathbb{R})/(\Gamma \cap G^0(\mathbb{R}))$ is open and closed in $G^0(\mathbb{R})/\Gamma$, and the latter is a finite union of translations of the former. It follows that (i) for $G$ is equivalent to (i) for $G^0$. On the other hand, in char 0, an element of finite order is always semisimple, and so if $G(\mathbb{Q})$ contains a non-trivial non-semisimple element $u$, its powers are also non-semisimple, but $u^m \in G^0$ for some $m$, showing that (ii) for $G$ is equivalent to (ii) for $G^0$. Assume from now on that $G$ is connected.

(i) $\Rightarrow$ (ii): Let $A : G \to G_m$ be a character defined over $\mathbb{Q}$, and let $v \in Q^\times$. By Lemma 51, $A(G(\mathbb{R})) \cdot v$ is closed in $\mathbb{R}$; on the other hand, if $A$ is non-trivial, this orbit contains at least $\mathbb{R}_{>0}$, but not the origin, hence it is not closed. Therefore, $A = 1$.

If one applies Lemma 51 to the representation of $G$ obtained by letting $G$ act on $M_n(\mathbb{C})$ by inner automorphisms, one sees that the conjugacy classes of elements of $G(\mathbb{Q})$ are closed. Now, let $1 \neq u \in G(\mathbb{Q})$ be a unipotent element. By a Theorem of Jacobson-Mostow (!!! add !!!), one can find a morphism $\sigma : SL_2 \to G$ which sends $u_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ to $u$. The conjugacy class of $u_0$ in $SL_2(\mathbb{R})$ contains the matrices
$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot u_0 \cdot \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2 \\ 0 & 1 \end{pmatrix}$$
therefore its closure contains 1. Therefore, the closure of the conjugacy class of $u$ in $G(\mathbb{R})$ also contains 1, contradiction.
(ii) ⇒ (i): First consider the case where $G$ has trivial center. Then $G$ is semisimple, and the adjoint representation of $\text{Ad} : G \to GL(g)$ is faithful. As it is defined over $\mathbb{Q}$, we may assume that $\Gamma$ is the stabilizer of a lattice $L$ in $g_{\mathbb{Q}}$. It is then enough to check that the condition (iv) (Mahler’s criterion) of Proposition 43 is fulfilled. First, since $G$ is semisimple, $|\det q| = 1$ for all $g \in G$. Write the characteristic polynomial of $x \in g$ as
\[
\det(ad\,x - T \cdot I) = (-T)^n + \sum P_i(x) \cdot T^{n-i}
\]
where $T$ is the variable. The $P_i(x)$ are then polynomials with rational coefficients, with respect to a base of $g$ formed of elements of $L$. Let $P = \sum P_i^2$. If $x \in L - \{0\}$, then $ad\,x$ is not nilpotent, since $g_{\mathbb{Q}}$ doesn’t contain nonzero nilpotent elements, hence $P(x) \neq 0$. But $P$ has rational coefficients, and can be written as $P' \cdot q$ where $P'$ has integral coefficients and $q \in \mathbb{Q}^\times$. Thus, there exists $c > 0$ such that if $x \in L - \{0\}$, $P(x) \geq c$. Now, if $(g_j)$ and $(v_j)$ are sequences of elements of $G(\mathbb{R})$ and $L$ such that $\text{Ad}\,g_j$ goes to $0$, we have
\[
P(v_j) = P(\text{Ad}\,g_j(v_j)) \rightarrow 0
\]
so that $v_j = 0$ for all $j$ large enough.

We now pass to the general case. Let $G^{ad} = G/Z(G)$, let $\Gamma^{ad}$ be an arithmetic subgroup of $G^{ad}$ containing $\pi(\Gamma)$ (exists, by Corollary 18 (3)). Let us show that $G^{ad}$ satisfies (ii). Since $(G^{ad})^0 = \pi(G^0)$, clearly $X((G^{ad})^0(\mathbb{Q})) = \{1\}$. Suppose that $G^{ad}(\mathbb{Q})$ contains a non-semisimple element $\neq 1$. Then it contains a unipotent element $\neq 1$, hence of infinite order, and $(G^{ad})^0(\mathbb{Q})$ also contains such $x \neq 1$ unipotent of infinite order. Then one must have $x \in H := D(G^{ad})^0$ an element of the derived group. Then $\log x$ is a nilpotent element of $h_{\mathbb{Q}}$. Since $\pi$ is an isogeny of $D\,G^0$ on $H$, it follows that the Lie algebra of $D\,G^0$ contains an element $x \neq 0$, rational over $\mathbb{Q}$, and nilpotent. Then $\exp x \in G(\mathbb{Q})$ is a unipotent element $\neq 1$, a contradiction.

By what we have already proven, $G^{ad}(\mathbb{R})/\Gamma^{ad}$ is compact. The image of $G(\mathbb{R})/\Gamma$ in it is closed, so it is enough to show that the natural map $G(\mathbb{R})/\Gamma \to G^{ad}(\mathbb{R})/\Gamma^{ad}$ is proper. This follows from Proposition 44. □

This proves (b). Now for (a).

First, we need to generalize the notion of a Siegel set.

**Definition 53.** Let $F$ be a subfield of $\mathbb{R}$, $G$ a reductive $F$-group, $P$ a minimal parabolic $F$-subgroup of $G^0$, $S$ a maximal split $F$-torus in $P$, $U$ the unipotent radical of $P$, $M$ the maximal anisotropic $F$-subgroup of the centralizer $Z(S)^0$ of $S$ in $G^0$, and $\Delta(F)$ the set of simple $F$-roots of $G$ w.r.t. $S$, for an order associated to $U$. Let $\mathfrak{f} \Delta := S^0(\mathbb{R})$. For any $t > 0$ we set
\[
\mathfrak{f}A_t := \{a \in F \ A \ | \ a^\alpha \leq t \ \alpha \in F \ \Delta\}
\]
Let $K$ be a maximal compact subgroup of $G(\mathbb{R})$. A **Siegel set** of $G(\mathbb{R})$ (w.r.t. $K, P, S$) is the product set
\[
\mathfrak{S} = \mathfrak{S}_{t,\omega} = K \cdot \mathfrak{f} \ A_t \cdot \omega
\]
where $\omega$ is a compact neighbourhood of $1$ in $M^0(\mathbb{R}) \cdot U(\mathbb{R})$.

We need the following theorem:
Theorem 54. ([1], Theorem 13.1) Let \( G \) be a reductive \( \mathbb{Q} \)-group, \( P \) a maximal parabolic subgroup of \( G \), \( S \) a maximal \( \mathbb{Q} \)-split torus of \( P \), and \( K \) a maximal compact subgroup of \( G(\mathbb{R}) \). Let \( \Gamma' \) be an arithmetic subgroup of \( G \). Then there exists a Siegel set (w.r.t. to \( K, P, S \)) \( \mathcal{S} \) and a finite subset \( C \) of \( G(\mathbb{Q}) \), such that
\[
G(\mathbb{R}) = \mathcal{S} \cdot C \cdot \Gamma
\]

Proof. We first show that if the Theorem holds for one choice of \( K \), then it holds for any other choice \( K' \). By (12.4 - complete!!!), there exists \( t' > 0 \) such that
\[
K \cdot P_t \subseteq K' \cdot P_{t'}
\]
Let \( \Gamma' = \Gamma \cap (\bigcap_{c \in C} c \Gamma c^{-1}) \). This is an arithmetic group, which satisfies
\[
\Gamma' \cdot C \subseteq C \cdot \Gamma
\]
By the compactness criterion (8.7 - !!!!add), there exist compact sets \( \alpha \subseteq M^0(\mathbb{R}) \) and \( \beta \subseteq U(\mathbb{R}) \) such that
\[
M^0(\mathbb{R}) = \alpha \cdot (\Gamma' \cap M), \quad U(\mathbb{R}) = \beta \cdot (\Gamma' \cap U)
\]
Using the fact that \( K' \cdot P_t^0 = K' \cdot M^0(\mathbb{R}) \cdot Q A_t \cdot U(\mathbb{R}) \), one sees that
\[
K' \cdot P_{t'} \cdot C \subseteq K' \cdot Q A_{t'} \cdot (M \cap \Gamma') \cdot U(\mathbb{R}) \cdot C = K' \cdot Q A_{t'} \cdot \alpha \cdot (M \cap \Gamma') \cdot U(\mathbb{R}) \cdot C \subseteq K' \cdot Q A_{t'} \cdot \alpha \cdot \beta \cdot \Gamma' \cdot C
\]
from which
\[
K \cdot P_{t} \cdot C \subseteq K' \cdot Q A_{t'} \cdot \omega' \cdot C \cdot \Gamma, \quad \omega' = \alpha \beta
\]
As \( K \) meets every connected component of \( G(\mathbb{R}) \), it suffices to show the theorem when \( G(\mathbb{C}) \) is connected. Suppose that \( G \hookrightarrow GL_n \).

Then (9.8, 9.9 - complete !!!) we know that given \( u \in GL_n(\mathbb{R}) \) such that \( u \cdot G(\mathbb{R}) \cdot u^{-1} \) is auto-adjoint, there exists a standard Siegel domain \( \mathcal{S} \) of \( GL_n(\mathbb{R}) \) and a finite subset \( B \) of \( GL_n(\mathbb{Q}) \) such that \( G(\mathbb{R}) = (u^{-1} \cdot \mathcal{S} \cdot B \cap G(\mathbb{R})) \cdot \Gamma \).

On the other hand, we know we can find \( u \in GL_n(\mathbb{R}) \) such that \( u \cdot G(\mathbb{R}) \cdot u^{-1} \) is auto-adjoint and “well placed” (by 11.25 - complete !!!)

Theorem 55. ([1], Theorem 13.5) With the same hypotheses as in Theorem 54, assume \( G \) is connected and \( S(\mathbb{R}) \) is stable under the Cartan involution associated with \( K \). Let \( u \in GL_n(\mathbb{R}) \) be such that \( u \cdot G(\mathbb{R}) \cdot u^{-1} \) is “well-placed”, \( b \in GL_n(\mathbb{Q}) \) and \( \mathcal{S}_0 \) a standard Siegel set of \( GL_n(\mathbb{R}) \). Then there exists a Siegel set \( \mathcal{S} \) of \( G(\mathbb{R}) \), w.r.t. \( K, P, S \) and a finite subset \( C \) of \( G(\mathbb{Q}) \) such that
\[
u^{-1} \cdot \mathcal{S}_0 \cdot b \cap G(\mathbb{R}) \subseteq \mathcal{S} \cdot C \cdot \Gamma
\]

Proof. (Complete later - if at all - this is heavy lifting)

Example 56. Let \( B \) be an indefinite quaternion algebra over \( \mathbb{Q} \) (i.e., such that \( B \otimes \mathbb{Q} \cong M_2(\mathbb{R}) \)). Let \( G \) be the algebraic group over \( \mathbb{Q} \) such that \( G(\mathbb{Q}) = \{ b \in B \mid \text{Nm}(b) = 1 \} \). (reduced norm). The choice of an isomorphism \( B \otimes \mathbb{Q} \to M_2(\mathbb{R}) \) determines an isomorphism \( G(\mathbb{R}) \to SL_2(\mathbb{R}) \), and hence an action of \( G(\mathbb{R}) \) on \( \mathcal{H} \). Let \( \Gamma \) be an arithmetic subgroup of \( G(\mathbb{Q}) \).
If $B$ is isomorphic to $M_2(\mathbb{Q})$, then $G$ is isomorphic to $SL_2$, which is semisimple, and so $\Gamma \backslash SL_2(\mathbb{R})$ (hence also $\Gamma \backslash H$) has finite volume. However, $SL_2(\mathbb{Q})$ contains a unipotent element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and so $\Gamma \backslash SL_2(\mathbb{R})$ is not compact.

If $B$ is not isomorphic to $M_2(\mathbb{Q})$, then it is a division algebra, and so $G(\mathbb{Q})$ contains no unipotent elements $\neq 1$ (otherwise $B^\times$ would contain a nilpotent element). Therefore, $\Gamma \backslash G(\mathbb{R})$ is compact. In this way, we get compact quotients $\Gamma \backslash H$ of $H$.

(*This is the idea of the JL transfer*)

**Definition 57.** Let $k \subseteq \mathbb{C}$ be a subfield. An automorphism of a $k$-vector space $V$ is said to be neat if its eigenvalues in $\mathbb{C}$ generate a torsion-free subgroup of $\mathbb{C}^\times$.

**Example 58.** A nontrivial automorphism of finite order is not neat.

**Definition 59.** Let $G$ be an algebraic $\mathbb{Q}$-group. We say $g \in G(\mathbb{Q})$ is neat if there exists a faithful representation $\rho : G \hookrightarrow GL_V$ for which $\rho(g)$ is neat.

**Remark 60.** If $g \in G(\mathbb{Q})$ is neat, then for any representation $\rho : G \rightarrow GL_V$ defined over a subfield of $\mathbb{C}$, $\rho(g)$ is neat. (all representations can be constructed from one faithful rep - check !?).

**Definition 61.** A subgroup of $G(\mathbb{Q})$ is neat if all its elements are neat.

**Proposition 62.** ([1], Proposition 17.4) Let $G$ be an algebraic group over $\mathbb{Q}$, and let $\Gamma$ be an arithmetic subgroup of $G(\mathbb{Q})$. Then $\Gamma$ contains a neat subgroup $\Gamma'$ of finite index. Moreover, $\Gamma'$ can be chosen to be defined by congruence conditions. (i.e. for some embedding $G \hookrightarrow GL_n$ and $N$, $\Gamma' = \{g \in \Gamma \mid g \equiv 1 \mod N\}$)

**Proof.** We may assume $G \subseteq GL_n$ and $\Gamma \subseteq GL_n(\mathbb{Z})$, by Corollary 18. Therefore, it is enough to prove the proposition for $\Gamma = GL_n(\mathbb{Z})$.

Let $M$ be the set of cyclotomic polynomials of degree $\leq n$, other than $T-1$. It is finite. Let $p$ be a prime not dividing any of the numbers $\{f(1)\}_{f \in M}$. We will show that the congruence subgroup

$$H = \{g \in GL_n(\mathbb{Z}) \mid g \equiv 1 \mod p\}$$

is neat.

Let $g \in H$, $(s_i)_{i=1}^n$ its eigenvalues, and $K$ the field extension of $\mathbb{Q}$ they generate. As the $s_i$ are roots of the characteristic polynomial of $g$, which is of degree $n$, with rational coefficients, $[\mathbb{Q}(s_i) : \mathbb{Q}] \leq n$. In particular, if $s_i$ is a root of unity, it is a root of a cyclotomic polynomial of degree $\leq n$.

Let $s_i$ be a root of unity. We will show that $s_i = 1$. Assume that this is not the case. Let $p$ be a prime of $K$ above $p$. The $s_i$ reduce mod $p$ to the eigenvalues of the reduction mod $p$ of $g$, which is 1. Therefore $s_i \equiv 1 \mod p$. Since $s$ is a root of unity of order at most $n$, there exists $f \in M$ such that $f(s_i) = 0$. As a consequence, $f(1) \equiv 0 \mod p$, hence also mod $p$, a contradiction. \hfill $\square$

**Remark 63.** This proposition can be generalized as follows - if $\text{char}(k) = 0$, $L$ is a subring of $K$ of finite type over $\mathbb{Z}$ and $\Gamma$ is a subgroup of $GL_n(L)$, then $\Gamma$ has a neat subgroup of finite index, which as a corollary also implies that any finite type subgroup of $GL_n(k)$ has a neat subgroup of finite index.
Definition 64. Let $H$ be a connected Lie group. A subgroup $\Gamma$ of $H$ is arithmetic if there exists an algebraic group $G$ over $\mathbb{Q}$, a surjective homomorphism $G(\mathbb{R})^+ \to H$ with compact kernel, and an arithmetic subgroup $\Gamma_0$ of $G(\mathbb{Q})$ such that $\Gamma_0 \cap G(\mathbb{R})^+$ maps onto $\Gamma$. (If $H$ is semisimple, we take $G$ to be semisimple).

Proposition 65. Let $H$ be a semisimple real Lie group that admits a faithful finite-dimensional representation. Every arithmetic subgroup $\Gamma$ of $H$ is discrete of finite covolume, and it contains a torsion-free subgroup of finite index.

Proof. Let $\alpha : G(\mathbb{R})^+ \to H$ and $\Gamma_0 \subseteq G(\mathbb{Q})$ be as in the definition of an arithmetic subgroup. $\ker \alpha$ is compact, hence $\alpha$ is proper, and in particular, closed. Because $\Gamma_0$ is discrete in $G(\mathbb{R})$, there exists an open $U \subseteq G(\mathbb{R})^+$ whose intersection with $\Gamma_0 \cdot \ker \alpha$ is exactly $\ker \alpha$.

Now, $\alpha(G(\mathbb{R})^+ \setminus U)$ is closed in $H$, and its complement intersects $\Gamma$ in $\{1\}$. (Wait! Why? If $h \notin \alpha(G(\mathbb{R})^+ \setminus U)$, either $h \notin \text{im}(\alpha)$ or $h \in \alpha(U)$. surjectivity shows that $h = \alpha(u)$, if $\alpha(u) \in \Gamma$, then by definition, there exists $v \in \Gamma_0 \cap G(\mathbb{R})^+$ such that $\alpha(u) = \alpha(v)$, hence $v^{-1}u \in \ker \alpha$, hence $u \in v \cdot \ker \alpha \subseteq \Gamma_0 \cdot \ker \alpha$. But $u \in U$, hence $u \in \ker \alpha$, so that $h = 1$!)

Therefore, $\Gamma$ is discrete in $H$. $\Gamma_0 \setminus G(\mathbb{R})^+$ maps onto $\Gamma \setminus H$, so applying Theorem 50 (a) to $G$ and $\Gamma_0$, we see that $\Gamma$ has finite covolume. (recall $G$ is semisimple).

Let $\Gamma_1$ be a neat subgroup of $\Gamma_0$ of finite index (exists by Proposition 62). Then $\alpha(\Gamma_1 \cap G(\mathbb{R})^+)$ has finite index in $\Gamma$, and its image under any faithful representation of $H$ is torsion-free. (by definition of neatness).

Remark 66. There are many non-arithmetic subgroups of $SL_2(\mathbb{R})$ of finite covolume. According to the Riemann mapping theorem, every compact Riemann surface of genus $g \geq 2$ is the quotient of $\mathcal{H}$ by a discrete subgroup of $PGL_2(\mathbb{R})^+$ acting freely on $\mathcal{H}$. Since there are continuous families of such surfaces, there are uncountable many discrete cocompact such subgroups (therefore also in $SL_2(\mathbb{R})$). However, in any connected real Lie group there are only countably many arithmetic subgroups (up to conjugacy - reference ??).

However, $SL_2$ is exceptional in that regard. By a theorem of Margulis, every discrete subgroup of finite covolume in a noncompact simple real Lie group $H$ is arithmetic unless $H$ is isogenous to $SO(1,n)$ or $SU(1,n)$. Note that $SL_2(\mathbb{R})$ is isogenous to $SO(1,2)$. (!!! Later - go back to Witte Morris !!!)

5. Algebraic Varieties VS Complex Manifolds

We would like to give our complex manifold an algebraic structure, so that we will be able to use it for rationality properties.

Now, we know that we can analytify a smooth algebraic variety.

Proposition 67. There is a unique functor $(V, \mathcal{O}_V) \mapsto (V^{an}, \mathcal{O}_{V^{an}})$ from nonsingular algebraic varieties over $\mathbb{C}$ to complex manifolds, with the following properties:

(a) $V = V^{an}$ as sets, every Zariski-open subset is open for the complex topology, and every regular function is holomorphic. (the identity map $V^{an} \to V$ is a morphism of $\mathbb{C}$-ringed spaces).

(b) If $V = \mathbb{A}^n$, then $V^{an} = \mathbb{C}^n$ with its natural structure as a complex manifold.

(c) If $\varphi : V \to W$ is etale, then $\varphi^{an} : V^{an} \to W^{an}$ is a local isomorphism.
Obviously, a regular map $\varphi: V \to W$ is determined by $\varphi^\text{an}: V^\text{an} \to W^\text{an}$, but not every holomorphic map $V^\text{an} \to W^\text{an}$ is regular. For example $z \mapsto e^z: \mathbb{C} \to \mathbb{C}$ is not.

Moreover, a complex manifold need not arise from an nonsingular algebraic variety, and two nonsingular algebraic varieties $V$ and $W$ can be isomorphic as complex manifolds without being isomorphic as algebraic varieties (Shafarevich gives the following example - let $C$ be a nonsingular projective cubic plane curve, $Q \in C$ a point, $B = C \setminus Q$ a noncomplete curve, $P \in B$. Let $X$ be the line bundle on $B$ corresponding to $P$ (as a divisor), $Y = B \times \mathbb{A}^1$ the trivial line bundle (corresponding to 0), then $X$ and $Y$ are not isomorphic, but $X^\text{an}$ and $Y^\text{an}$ are isomorphic, basically because $[P] - [Q]$ is not the divisor of any rational function, as $C$ is cubic, but it is the divisor of a meromorphic function). In other words, $V \sim V^\text{an}$ is faithful, but is neither full nor essentially surjective on objects.

Remark 68. The analytification functor can be extended to all algebraic varieties, once one has the notion of “complex manifold with singularities”. This is a complex analytic space. For holomorphic functions $f_1, \ldots, f_r$ on a connected open subset $U$ of $\mathbb{C}^n$, let $V(f_1, \ldots, f_r)$ be the set of common zeros. It has a natural structure of a ringed space, and now glue them together.

Here are two necessary conditions for a complex manifold to arise from an algebraic variety.

(a) It must be possible to embed $M$ as an open submanifold of a compact complex manifold $M^*$ in such a way that the boundary $M^* \setminus M$ is a finite union of manifolds of dimension $\dim M - 1$.

(b) If $M$ is compact, then the field of meromorphic functions on $M$ must have transcendence degree $\dim M$ over $\mathbb{C}$.

The necessity of (a) follows from Hironaka’s theorem on the resolution of singularities, which shows that every nonsingular variety $V$ can be embedded as an open subvariety of a complete nonsingular variety $V^*$ in such a way that the boundary $V^* \setminus V$ is a normal crossings divisor. (Basically, this is his Main Theorem II)

The necessity of (b) comes from the fact that when $V$ is complete and nonsingular, the field of meromorphic functions on $V^\text{an}$ coincides with the field of rational functions on $V$ (Rough sketch - since $V^\text{an}$ is compact, it’s enough to show that if $f$ is meromorphic and algebraic over $\mathbb{C}(V)$ then it is rational. algebraicity $\Rightarrow$ rational functions, discard poles, assume they are regular, then as local rings are UFD, hence integrally closed in field of fractions, $f$ is holomorphic. look at polynomials with two variables, project. ).

Theorem 69. ([3], VIII, 3.1, Theorem 1) If $V$ is a complete algebraic variety over $\mathbb{C}$ then a meromorphic function on the complex manifold $V^\text{an}$ is a rational function on $V$.

(Do I want to add the proof?)

Nonetheless, we have one positive result.

Restricting analytification to the category of projective non-singular curves over $\mathbb{C}$, with image being compact Riemann surfaces, one obtains an equivalence of categories.
Since the proper Zariski-closed subsets of algebraic curves are the finite subsets, we see that for Riemann surfaces, condition (a) is also sufficient: a Riemann surface is algebraic iff it is possible to embed $M$ in a compact Riemann surface $M^*$ in such a way that $M^* \setminus M$ is finite.

By the maximum modulus principle, any holomorphic function on a connected compact Riemann surface is constant. Therefore, if $M$ is algebraic, then every bounded holomorphic function on $M$ is constant. We conclude that $\mathcal{H}$ does not arise from an algebraic curve, because $z \mapsto \frac{z^2 + 1}{z + i}$ is bounded, holomorphic, and non-constant.

For all lattices $\Lambda$ in $\mathbb{C}$, the Weierstrass $\wp$ function, and its derivative $\wp'$ embed $\mathbb{C}/\Lambda$ into $\mathbb{P}^2(\mathbb{C})$ (as an elliptic curve). However, for a lattice $\Lambda \subseteq \mathbb{C}^2$, the field of meromorphic functions on $\mathbb{C}^2/\Lambda$ will usually have transcendence degree $< 2$, and so $\mathbb{C}^2/\Lambda$ is not an algebraic variety. (it is algebraic precisely when it admits a Riemann form. Those that admit a Riemann form are a proper closed subset of the moduli space. The lattice generated by $(1, 0), (i, 0), (0, 1), (\sqrt{2}, i)$ does not admit a Riemann form.)

For quotients of $\mathbb{C}^g$ by a lattice $\Lambda$, condition (b) is sufficient for algebraicity. ([7], p. 35, Corollary)

However, if we restrict ourselves to projective varieties and manifolds, we have the following theorem.

**Theorem 70.** (Chow) Every projective complex analytic space has a unique structure of a projective algebraic variety, and every holomorphic map of projective complex analytic spaces is regular for these structures. The algebraic variety attached to a complex manifold is nonsingular. ([3], VIII, 3.1, Theorems 2,3) (?? I think I will not add proof - this is a quite standard result in algebraic geometry ??)

In other words, the functor $V \mapsto V^{\text{an}}$ is an equivalence of categories between projective algebraic varieties and projective complex analytic spaces, under which nonsingular algebraic varieties map to complex manifolds.

6. **The Theorem of Baily and Borel**

The actual reason for our interest in arithmetic subgroups lies in the following deep and incredible result.

**Theorem 71.** (!!! add reference !!!) Let $\Gamma \backslash D$ be the quotient of a Hermitian Symmetric Domain $D$ by a torsion-free arithmetic subgroup $\Gamma$ of $\text{Hol}(D)^+$. Then $\Gamma \backslash D$ has a canonical realization as a Zariski-open subset of a projective algebraic variety - $(\Gamma \backslash D)^*$. In particular, it has a canonical structure as an algebraic variety.

Recall briefly how one does it for the modular curves.

6.1. **The Harish-Chandra Embedding.** Recall that the upper half plane can be transformed to the unit disc (the Poincare model), using the Cayley transform $z \mapsto \frac{z + i}{z - i}$. This yields a bounded symmetric domain as a model.

In general, we have 4 classical families of Hermitian symmetric domains, and each have a classic bounded and unbounded models.
Example 72. The four families of unbounded classical domains are:
1. The Hermitian upper half space. \( I_{p,q} := \{(Z,U) \in M_q(\mathbb{C}) \times M_{(p-q) \times q}(\mathbb{C}) \mid -i \cdot (Z - Z^*) - U^*U > 0\} \).
2. The quaternionic upper half space \( II_n := \{Z \in M_n(\mathbb{H}) \mid i \cdot Z - Z^* \cdot i > 0\} \).
3. The Siegel upper half space

\[ \text{III}_n := \mathcal{H}_n = \{Z = X + iY \mid \Im(Y) > 0, Z^t = Z \in M_n(\mathbb{C})\} = \{Z \in M_n(\mathbb{C}) \mid Z^t = Z, -i \cdot (Z - Z^*) > 0\} \]

, a straight-forward generalization of \( \mathcal{H} = \mathcal{H}_n \).
4. \( IV_n := \{(z, u) \in \mathbb{C} \times \mathbb{C}^n \mid -i \cdot (z - \bar{z}) - |u|^2 > 0\} \).

Which have the following realization as bounded domains: (Exercise to verify)
1. \( I_{p,q} = \{Z \in M_{p \times q}(\mathbb{C}) \mid I_q - Z^*Z > 0\} \cong U(p,q)/U(p) \times U(q) \).
2. \( II_n = \{Z \in M_n(\mathbb{C}) \mid Z^t = -Z, I_n - Z^*Z > 0\} \cong O^*(2n)/U(n) \). Here \( O^*(2n) = \{g \in GL_n(\mathbb{H}) \mid g^*(i \cdot I_n)g = i \cdot I_n\} \) is the quaternionic skew-Hermitian group.
3. \( III_n = D_n = \{Z \in M_n(\mathbb{C}) \mid Z^t = Z, I_n - Z^*Z > 0\} \cong Sp_{2n}(\mathbb{R})/U(n) \). (Here we have an analogue of the Cayley transform: \( Z \mapsto (Z - i \cdot I_g) \cdot (Z + i \cdot I_g)^{-1}\)).
4. \( IV_n = \{z \in \mathbb{C}^n \mid |z|^2 < \frac{1+|i\sum \bar{z}^2|}{2} < 1\} \cong O(n,2)/O(n) \times O(2) \).

One can see the isomorphism by identifying the isotropy groups in the unbounded case, and see that they are conjugate to these.

It turns out that this can be done more generally. This was done by Harish-Chandra, and Baily and Borel use this embedding to construct the boundary components.

We have the following theorem:

Theorem 73. Let \( G \) be a connected reductive algebraic group over \( \mathbb{R} \), such that \( X(G)_R = \{1\} \). Let \( K \) a maximal compact subgroup of \( G(\mathbb{R})^+ \). Let \( \mathfrak{g}, \mathfrak{k} \) be their Lie algebras. Then there exist abelian subalgebras of \( \mathfrak{g}_C, \mathfrak{p}^\pm \), normalized by \( \mathfrak{k}_C \), such that \( \mathfrak{g}_C = \mathfrak{k}_C \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_- \). Moreover, if \( \mathfrak{p}^\pm = \exp(\mathfrak{p}^\pm) \), then \( G(\mathbb{R})^+ \subset P^- \cdot K(\mathbb{C}) \cdot P^+ \). The map \( \zeta : G(\mathbb{R})^+ \rightarrow \mathfrak{p}^+ \) induced by the decomposition and log, induces an isomorphism of \( K \backslash G(\mathbb{R})^+ \) onto \( \zeta(G) \), and this is a bounded domain in \( \mathfrak{p}^+ \).

(!!! later - add explanation for this decomposition. It is not difficult)

Example 74. Consider the case \( G = Sp_{2,2} \). Recall we considered earlier the Siegel upper half space \( \mathcal{H}_g = \{Z = X + iY \mid \Im(Y) > 0, Z^t = Z \in M_g(\mathbb{C})\} \), as a natural generalization of the upper half plane \( \mathcal{H} = \mathcal{H}_1 \). We shall look instead in its bounded model \( III_g \). Recall also that

\[
G = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A^tC = C^tA, B^tD = D^tB, A^tD - C^tB = I_g, D^tA - B^tC = I_g \right\}
\]

so that the stabilizer of \( Z = i \cdot I_g \) (still in the unbounded model) has \( C = -B, D = A \), hence \( B^tA = A^tB \) and \( A^tA + B^tB = I_g \).

Conjugating by \( \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \) leads to

\[
\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \cdot \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ i & -i & -i & i \end{pmatrix} \cdot \begin{pmatrix} A + iB & B - iA \\ A - iB & B + iA \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} A + iB & 0 \\ 0 & A - iB \end{pmatrix}
\]

Note that 

\[(A + iB)(A - iB) = A^tA + iB^tA - iA^tB + B^tB = I_g, \text{ hence } A - iB = ((A + iB)^t)^{-1}.\]

Moreover, \(A - iB = (A + iB)\), so that \(A + iB \in U(g)\). Thus, in the bounded model, the maximal compact is 

\[K = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \mid A \in U(g) \right\} \cong U(g)\]

with Lie algebra 

\[\mathfrak{k} = \left\{ \begin{pmatrix} X & 0 \\ 0 & -X^t \end{pmatrix} \mid X \in u(g) \right\} \cong u(g)\]

so that its orthogonal complement with respect to the Killing form \(B(X, Y) = \text{tr}(X \cdot Y)\) is 

\[\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in g \mid AX - DX^t = 0 \quad \forall X \right\}\]

and so 

\[\mathfrak{p}^\perp = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \mid B = B^t \right\}, \mathfrak{p}^\perp = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \mid C = C^t \right\}\]

(Why???? Complete here)

Decomposing according to the maximal Cartan subalgebra of \(\mathfrak{k}\), we see that 

\[\mathfrak{p}^\perp = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \mid B = B^t \right\}, \mathfrak{p}^\perp = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \mid C = C^t \right\}\]

Also, note that 

\[K(\mathbb{C}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \mid A \in GL_g(\mathbb{C}) \right\} \cong GL_g(\mathbb{C})\]

Now, given \(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{R})\), we can try to find a decomposition (that we know exists) 

\[\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_g & X \\ 0 & I_g \end{pmatrix} \begin{pmatrix} H & 0 \\ 0 & (H^t)^{-1} \end{pmatrix} \begin{pmatrix} I_g & 0 \\ Y & I_g \end{pmatrix}\]

with some \(X, Y, H \in GL_g(\mathbb{C})\). Evaluating, we get 

\[\begin{pmatrix} H & X \cdot (H^t)^{-1} \\ 0 & (H^t)^{-1} \end{pmatrix} \begin{pmatrix} I_g & 0 \\ Y & I_g \end{pmatrix} = \begin{pmatrix} H + X \cdot (H^t)^{-1} \cdot Y \cdot (H^t)^{-1} \\ (H^t)^{-1} \cdot Y \cdot (H^t)^{-1} \end{pmatrix}\]

so that \((H^t)^{-1} = D\), hence \(H = (D^t)^{-1}\), \(D \cdot Y = C\), so that \(Y = D^{-1} \cdot C\), \(X \cdot D = B\), so that \(X = B \cdot D^{-1}\), and we need also that 

\[A = H + X \cdot D \cdot Y = (D^t)^{-1} + B \cdot D^{-1} \cdot C\]

i.e. 

\[D^t \cdot A = I_g + D^t BD^{-1} C\]

But we know that 

\[D^t \cdot A = I_g + B^t C = I_g + B^t DD^{-1} C = I_g + D^t BD^{-1} C\]

from the conditions of \(Sp_{2g}(\mathbb{R})\). Great.
So our map $\zeta : Sp_{2g}(\mathbb{R}) \to p^+$ is given (on the dense open set where $D$ is invertible) by
$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \mapsto \begin{pmatrix}
0 & \log(B \cdot D^{-1}) \\
0 & 0
\end{pmatrix}
$$
Note that $B \cdot D^{-1}$ is precisely evaluation at 0, which identified $Sp_{2g}(\mathbb{R})/K$ with the bounded model.

gist of the proof - add “rational boundary components” - these are obtained by looking at the Harish-Chandra boundary components, and describing rationality. This is endowed with the Satake topology. Take automorphic forms of sufficiently high weight and embed the quotient as a closed subvariety of projective space, and one gets the result.

!!!! Maybe give some more details and examples !!!!

Sketch of the proof:
1. We have the natural compactification of our Hermitian symmetric domain, using Harish-Chandra embedding. This gives rise to boundary components coming from the maximal parabolic subgroups of $G_\mathbb{R}$.

2. Identify the “rational boundary components” - those coming from the rational maximal parabolic subgroups (those for which the $\Gamma$-points are discrete and cocompact in the unipotent part). It turns out these are all of codimension at least 2 (usually).

3. Show that the closure of the Siegel domain (over $\mathbb{Q}$) used to cover $X$ (i.e. there exists a finite $C \subseteq G(\mathbb{Q})$ with $\Theta \cdot C = X$), is contained in the union of the standard rational boundary components, and its interection with them is always a Siegel domain in them.

4. Let $X^\ast$ be the union of $X$ and the rational boundary components. Then there is a holomorphic action of $G$ on $X$, and there is a fundamental set $\Omega = \text{im}(\Theta \cdot C)$ for $\Gamma$ in $X$ such that $\overline{\Omega} \cdot \Gamma = X^\ast$. Equip it with the Satake topology - a fundamental set of nbs for $x \in X^\ast$ is just the $U \subseteq X^\ast$ such that $U \cdot \gamma = U$ for all $\gamma \in \Gamma_x$ and $U \cdot \gamma \cap \overline{\Omega}$ is a nbhd of $x \cdot \gamma$ in $\overline{\Omega}$.

5. We can quotient by $\Gamma$! Have to check many details - that all the properties are inherited by boundary subspaces, etc. To give a more precise description of the nbds of a point in $X^\ast = \Gamma \backslash X^\ast$, use “truncated Siegel domains”.

6. Now that we have a compactification, we would like to construct automorphic forms, to provide an embedding into projective space. This is done by considering
$$
pf(g) = \sum_{\gamma \in \Gamma} f(g \cdot \gamma)
$$
where $f$ is some vector-valued function on $G_\mathbb{R}$ with some finiteness properties - this is a Poincare series.

For functions $F : D \to V$, where $D$ is the bounded realization, can associate $f : G(\mathbb{R}) \to V$ by $f(g) = \mu_r(o,g) \cdot F(\zeta(g)) = \rho(e^{io} \cdot g) \cdot F(\zeta(g))$, where $\rho$ is the representation on $V$.

Then one shows that the functional determinant $g \mapsto J(0, g) \in L^1(G(\mathbb{R}))$. Recall this is the determinant of the differential (Jacobian) of the automorphism induced by $g$ at 0.
More generally, if \( \varphi : D \to V \) is a polynomial mapping, and \( P_{\varphi}(x) = \sum_{\gamma \in \Gamma} J(x, \gamma) \varphi(x \cdot \gamma) \), so if \( f(g) = J(0, g)^a \cdot \varphi(\zeta(g)) \), then
\[
p_f(g) = J(0, g)^a \cdot P(\zeta(g))
\]

Then \( p_f \) is a Poincare series, it conv. absolutely anf uniformly, bla. Finally, if \( \Gamma_\infty \) is a subgroup of finite index in \( \Gamma \cap P \), one can consider the Eisenstein series \( E_f(g) = \sum_{\gamma \in \Gamma/\Gamma_\infty} f(g \cdot \gamma) \), where \( f : G(\mathbb{R}) \to V \) is a function with \( f(g \cdot p) = f(g) \cdot |\chi_p(g)| \).

Then, as \( \chi_s(\Gamma) \in \{\pm 1\} \), \( f \) is right invariant under \( \Gamma_\infty \), so the summation makes sense, and \( E_f \) is right-invariant under \( \Gamma \).

This converges for \( s \) with \( \Re(s_0) > e_\alpha \). This can be generalized to the case where \( f \) is not invariant, but only “almost” - the ratio is bounded uniformly on compact sets.

Introduce a generalization of both - Poincare-Eisenstein series:

For \( f : G(\mathbb{R}) \to V \) such that \( ||f(h \cdot c)|| = ||f(h) \cdot \Delta(c, s)|| \) for all \( c \in B(\mathbb{R}) \cap G(\mathbb{R}) \), \( h \in G(\mathbb{R}) \), let \( \Gamma_0 = \Gamma_\infty \cap B \). Then
\[
E_f(h) = \sum_{\gamma \in \Gamma/\Gamma_0} f(h \cdot \gamma)
\]

Thus we can consider \((\sigma_b : S_b \to D_b \text{ is the projection})\)
\[
E(x) = \sum_{\gamma \in \Gamma/\Gamma_0} \varphi(\sigma_b(x \cdot \gamma)) \cdot J_F(x, \gamma)^l
\]

One then shows that for suitable \( l \)'s (all positive multiples of some \( l_0 \)), the series \( E \circ g \) converges... (automorphic form of weight \( l \) for \( g^{-1} \Gamma g \)).

This represents a holomorphic cross-section of a complex line bundle on \( X \). One then studies their behaviour near boundary components in \( X^* \).

Continue to define integral automorphic forms - (can be extended to the boundary, and be an automorphic form for \( \Gamma(F) \)) there.

One shows that we can separate components using these forms, and hence that one can take enough that do not vanish simultaneously.

Remark 75. (a) The theorem also holds when \( \Gamma \) has torsion. Then \( \Gamma \setminus D \) is a normal complex analytic space (rather than a manifold) and \( \text{ti} \) has the structure of a normal algebraic variety (rather than nonsingular).

(b) The variety \((\Gamma \setminus D)^* \) is usually very singular. The boundary \( \partial(\Gamma \setminus D) \) has codimension \( \geq 2 \), provided \( PGL_2 \) is not a quotient of the \( \mathbb{Q} \)-group giving rise to \( \Gamma \).

(c) The variety \((\Gamma \setminus D)^* \) is constructed as \( \text{Proj} (\bigoplus_{n=0}^\infty A_n) \), where \( A_n \) is the vector space of automorphic forms for the \( n \)th power of the canonical automorphy factor. It follows that, if \( PGL_2 \) is not a quotient of \( G \), then
\[
(\Gamma \setminus D)^* = \text{Proj} \left( \bigoplus_{n \geq 0} H^0(\Gamma \setminus D, \omega^n) \right)
\]

where \( \omega \) is the sheaf of algebraic differentials of maximum degree on \( \Gamma \setminus D \).

Without the condition on \( G \), there is a similar description in terms of differentials with logarithmic poles.
(d) When $\Gamma \backslash D$ is compact, the theorem follows from the Kodaira embedding theorem. This can be extended to those having boundary of dimension 0.

**Definition 76.** The algebraic variety $\Gamma \backslash D$ arising as in the theorem is called a locally symmetric variety.

### 7. The Theorem of Borel

**Theorem 77.** Let $\Gamma \backslash D$ be the quotient of a Hermitian Symmetric Domain by a torsion-free arithmetic subgroup $\Gamma$ of $\text{Hol}(D)^+$, and let $V$ be a non-singular algebraic variety over $\mathbb{C}$. Then every holomorphic map $f : V^{an} \to D(\Gamma)^{an}$ is regular.

Let $(\Gamma \backslash D)^*$ be the Bailey-Borel compactification. The key step in Borel’s proof is the following result:

**Lemma 78.** Let $D^\times$ denote the punctured unit disk $\{z \mid 0 < |z| < 1\}$. Then every holomorphic map $(D^\times)^r \times D^s \to \Gamma \backslash D$ extends to a holomorphic map $D^{r+s} \to (\Gamma \backslash D)^*$ of complex spaces.

This is a generalization of the big Picard theorem - if a function $f$ has an essential singularity at $p \in \mathbb{C}$, then on any open disk containing $p$, $f$ takes every complex value except possibly one.

Therefore, if a holomorphic function on $D^\times$ misses two values it must have at worst a pole at 0, and so extends to a holomorphic function $D \to \mathbb{P}^1(\mathbb{C})$.

This can be restated as follows:

Every holomorphic function from $D^\times$ to $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ extends to a holomorphic function from $D$ to the natural compactification $\mathbb{P}^1(\mathbb{C})$.

**Proof.** of the Theorem, using the Lemma. According to Hironaka’s theorem on the resolution of singularities, we can realize $V$ as an open subvariety of a projective nonsingular variety $V^*$ such that $V^* \backslash V$ is a normal crossings divisor. This means that, locally for the complex topology, the inclusion $V \hookrightarrow V^*$ is of the form $(D^\times)^r \times D^s \hookrightarrow D^{r+s}$ (does it? yes, it does - by definition it is etale locally the intersection of coordinate hyperplanes. e.g. $xy = 0$) Therefore, the Lemma shows that $f : V^{an} \to (\Gamma \backslash D)^{an}$ extends to a holomorphic map $(V^{*})^{an} \to (\Gamma \backslash D)^*$, which is regular by Chow’s theorem.

**Corollary 79.** The structure of an algebraic variety on $\Gamma \backslash D$ is unique.

**Proof.** Let $D(\Gamma)$ be the canonical algebraic structure on $\Gamma \backslash D$ provided by Bailey-Borel, and suppose $\Gamma \backslash D = V^{an}$ for some algebraic variety $V$. Then the identity map $f : V \to D(\Gamma)$ is a regular (because of Borel’s theorem) bijective map of nonsingular varieties, therefore an isomorphism.

The proof of the theorem shows that for any compactification with normal crossings boundary divisor, $(\Gamma \backslash D)^\dagger$, there is a unique regular map $(\Gamma \backslash D)^\dagger \to (\Gamma \backslash D)^*$ making all commute. For this reason the Bailey-Borel compactification is also called the minimal compactification.
Definition 81. A semisimple group $G$ over $\mathbb{Q}$ is said to be of compact type if $G(\mathbb{R})$ is compact, and it is of noncompact type if it does not contain a nontrivial normal subgroup of compact type.

Lemma 82. Let $G$ be a semisimple group. Then there exist simple groups $\{G_i\}_{i=1}^n$ and an isogeny $\varphi : \prod_{i=1}^r G_i \to G$.

If all the $G_i(\mathbb{R})$ are compact, then clearly $G$ is of compact type. If no $G_i(\mathbb{R})$ is compact, then $G$ is of noncompact type (indeed, any normal subgroup is an almost direct product of these simple ones). In particular, a simply connected or adjoint group is of noncompact type iff it has no simple factor of compact type.

Theorem 83. (Borel Density Theorem) Let $G$ be a semisimple group over $\mathbb{Q}$ of noncompact type. Then every arithmetic subgroup $\Gamma$ of $G(\mathbb{Q})$ is Zariski-dense in $G$.

Proof. (!!! add later - from Borel !!!) 

Corollary 84. Let $G$ be a semisimple group over $\mathbb{Q}$ of noncompact type. Let $Z$ be its center. The centralizer in $G(\mathbb{R})$ of any arithmetic subgroup $\Gamma$ of $G(\mathbb{Q})$ is $Z(\mathbb{R})$.

Proof. By the theorem, the centralizer of $\Gamma$ in $G(\mathbb{C})$ is the same as the centralizer of $G$, i.e. $Z(\mathbb{C})$, and $Z(\mathbb{R}) = Z(\mathbb{C}) \cap G(\mathbb{R})$. □

Theorem 85. Let $D$ be a Hermitian Symmetric Domain. Let $\Gamma$ be a torsion-free arithmetic subgroup of $\text{Hol}(D)^+$. Then $\Gamma \backslash D$ has only finitely many automorphisms (as a complex manifold).

Proof. As $\Gamma$ is torsion-free, $D$ is the universal covering space for $\Gamma \backslash D$, and $\Gamma$ is the group of covering transformations. An automorphism $\alpha : \Gamma \backslash D \to \Gamma \backslash D$ lifts to an automorphism $\tilde{\alpha} : D \to D$. For all $\gamma \in \Gamma$, the map $\tilde{\alpha} \gamma \tilde{\alpha}^{-1}$ is a covering transformation, hence lies in $\Gamma$. Conversely, an automorphism of $D$ normalizing $\Gamma$ defines an automorphism of $\Gamma \backslash D$. Thus $\text{Aut}(\Gamma \backslash D) = N(\Gamma)/C(\Gamma)$. (normalizer and centralizer, resp. in $\text{Aut}(D)$).

By assumption, there exists a semisimple algebraic group $G$ over $\mathbb{Q}$, a surjective homomorphism $G(\mathbb{R})^+ \to \text{Hol}(D)^+$ with compact kernel, and an arithmetic subgroup $\Gamma_0$ of $G(\mathbb{Q})$ such that $\Gamma_0 \cap G(\mathbb{R})^+$ maps onto $\Gamma$ with finite kernel. We may discard any compact isogeny factors of $G$, and so suppose that $G$ is of noncompact type. (Wait!!! Why ??? it seems that compact subgroups will give trivial images... Check!!! How does 4.7 (a) $\iff$ (c) prove it?) Let $N^+$ be the identity component of $N(\Gamma)$. $\Gamma$ is discrete, hence $N^+$ acts trivially on it. (connected image), hence $N^+$ is contained in the (finite) center of $G(\mathbb{R})$ (by the corollary). Therefore, $N(\Gamma)$ is discrete. Because $\Gamma \backslash \text{Aut}(D)$ has finite volume, this implies that $\Gamma$ is of finite index in $N(\Gamma)$ (Witte-Morris !!! complete).
Alternate proof: (geometric, when $\Gamma$ is neat) By (Mumford!!!) $\Gamma \setminus D$ is an algebraic variety of logarithmic general type, which implies (Iitaka ??!!!) that its automorphism group is finite.

Remark 86. We have required $\Gamma$ to be torsion-free all along. In particular, we disallowed $\Gamma(1) \setminus \mathcal{H}$. For an arithmetic subgroup with torsion, the algebraic variety $\Gamma \setminus D$ may be singular and Borel's theorem fails.

REFERENCES