Shimura Varieties Seminar: More examples

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Reminders

- Choosing an endomorphism J such that $J^2 = -I$ is like defining "multiplication by i" on the underlying real vector space.
- $\mathbb{C}$ acts by $\lambda \cdot v := h(\lambda)v$.
- Must be compatible with some Riemann form if we want polarizations..
    - Riemann forms are *alternating*. With complex structure satisfy $\psi(iu, iv) = \psi(u, v)$.
    - Associated Hermitian form $i\psi(u, v) + i\psi(u, Jv) > 0$.

Exercise

Fix a standard symplectic form ψ.

$$G := \mathsf{GSp}_{2g}(\mathbb{R}).$$

Show that the standard Siegal modular variety

$$\mathcal{M} := \{A + Bi \in \mathrm{Sym}_g(\mathbb{C}) : B > 0\}$$

is identified with the description as a $G(\mathbb{R})$-conjugacy class of homomorphisms $h \colon \mathbb{S} \to G_{\mathbb{R}}$.

D := $G(\mathbb{R})$-conjugacy class of homomorphisms $h\colon \mathbb{S} \to G_{\mathbb{R}}$.

From last time:

$$X^+ = \{J \in Sp_{2g}(\mathbb{R}) : J^2 = -1, \psi_J(u,v) := \psi(u,Jv) > 0\}$$

was identified with D via

$$J \mapsto (\text{class of } h_J\colon (a+bi) \mapsto a+bJ) \in D$$

Exercise

(Note that the choice of ψ defining X is unimportant, as up to change of basis we may assume that ψ is the standard symplectic form.)

The "enlightened solution".
Since $\mathcal{M}/\Gamma(N)$ and $D/K(N)$ both parametrize abelian varieties with level N structure for all N, the moduli functors are represented by the same variety.

The "low brow" solution:
Given $A + Bi \in \mathcal{M}$, we distinguish $J \in X^+(\mathbb{R})$ whose centralizer is identified with the stabilizer of this point. By transitivity, we may assume that $A + Bi = iI$. We see that the stabilizer of this point is given by

$$\begin{bmatrix} X & Y \\ -Y & X. \end{bmatrix}$$

(To see this, solve the equation $iI = (Xi + Y)(W + iZ)^{-1}$ and remember X, Y, Z, W are real.)

We have

$$\mathrm{Stab}(\mathrm{Sp}_2\, g(\mathbb{R}), iI) = \begin{bmatrix} X & Y \\ -Y & X. \end{bmatrix} \cong \mathrm{U}_n(\mathbb{R})$$

This is exactly the centralizer of the homomorphism

$$h\colon x + iy \mapsto \begin{bmatrix} xI & yI \\ -yI & xI \end{bmatrix}.$$

Note that the centre of $\mathrm{U}(n)$ is the $\mathrm{U}(1)$ in the image of h. In this case, we set $J := h(i)$. $\square$

The unitary Shimura variety

Define $I_{a,b} := \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$ and define

$$\mathrm{U}_{a,b} := \{g \in \mathrm{GL}_{a+b} : g^* I_{a,b} g = I_{a,b}\}.$$

Denote the space of $a \times b$ matrices by $M_{a,b}$. We act on the space

$$D_{a,b} := \{U \in M_{a,b}(\mathbb{C}) : (\bar{U}, 1) I_{a,b} (U, 1)^T = U^* U - I_b < 0\}.$$

With

$$g := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{U}_{a,b}, \text{by}$$

$$gU := (AU + B)(CU + D)^{-1}.$$

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The stabilizer of $0 \in D_{a,b}$ is defined by $B = 0$ (implying $C = 0$), so

$$\mathsf{Stab}(\mathrm{U}_{a,b}, 0) := \mathrm{U}_a \times U_b \hookrightarrow \mathrm{U}_{a,b}\,.$$

This is exactly the centralizer of

$$h_0 \colon x + iy \mapsto (x - iy)I_a \oplus (x + iy)I_b.$$

Important point!: We can't quite say $(\mathrm{U}_{a,b}, D_{a,b})$ defines a Shimura datum. The reason being we have $h \colon U_1 \to \mathrm{U}_{a,b}$, instead of $h \colon \mathbb{S} \to \mathrm{U}_{a,b}$.

Instead: Consider $h_0 \colon \mathbb{S} \to \mathrm{GU}_{a,b}$ defined by

$$h_0(x + iy) \mapsto (x - iy)I_a \oplus (x + iy)I_b.$$

Checking (remembering) the axioms

Proposition

The Axioms SV1-SV3 are invariant under conjugation. In other words, it suffices to check the axioms for a single $h \in X$.

Proofs follow:

SV1

For all $h \in X$, the Hodge structure on $Lie(G_{\mathbb{R}})$ defined by $Ad \circ h$ is of type
$$\{(-1,1),(0,0),(1,-1)\}.$$

Proof: The Hodge structure defined by a representation $h \colon \mathbb{S} \to V_{\mathbb{R}}$ is defined by the invariant subspace decomposition for the action of the torus $\mathbb{S}$. Conjugation does not change factors of the decomposition or the weights. $\square$

SV2

For all $h \in X$, $\mathrm{ad}(h(i))$ is a Cartan involution of $G^{\mathrm{ad}}_{\mathbb{R}}$

(An involution is Cartan if $G^{\theta}(\mathbb{R})$ is compact.)

Proof: If $g \in G(\mathbb{R})$, the fixed locus of

$$\mathrm{ad}(gh(i)g^{-1}) = \mathrm{ad}(g) \cdot \mathrm{ad}(h(i)) \cdot \mathrm{ad}(g)^{-1}$$

is also compact.

SV3

G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

Proof: If $G \cong G_0 \times G_1$, conjugation is covariant with respect to the projection maps. If $\pi_0(h), \pi_1(h)$ are both nontrivial, this is true after conjugation too.

SV4-SV6

SV4

The weight homomorphism $w_X : \mathbb{G}_m \to G_{\mathbb{R}}$ is defined over $\mathbb{Q}$.

SV5

The group $Z(\mathbb{Q})$ is discrete in $A(\mathbb{A}_f)$.

SV6

The torus Z° split over a *CM*-field.

These aren't really element specific anyway.

Checking the axioms

Proposition

The axioms SV1-SV4, SV6 hold for $(\mathrm{GU}_{a,b}, D_{a,b})$. SV5 does not hold.

It suffices to check SV1-SV3 for h_0.

(SV1) Note $h_0(z) = zI_a \oplus \bar{z}I_b$, the adjoint action is given by

$$h_0(z) \cdot \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \begin{bmatrix} A & B \cdot \left(\frac{z}{\bar{z}}\right) \\ C \cdot \left(\frac{\bar{z}}{z}\right) & D \end{bmatrix}$$

which clearly has the correct weights.

(SV2) We have computed that the stabilizer of $h_0(i)$ is $\mathsf{U}_a \times \mathsf{U}_b$, which is compact.

(SV3) The group $\mathsf{GU}^{\mathrm{ad}} = \mathsf{SU}_{a,b}$ is simple.

For the remaining axioms, we need to view $\mathsf{GU}_{a,b}$ as being given by algebraic equations in the affine space $\mathbb{A}^{2(a+b)}$.

(SV4) The weight homomorphism is just the diagonal map $z \mapsto zI$.

(SV5) The centre as a model in $M_{a+b}(\mathbb{C})$ is defined by $\{zI : z \in \mathbb{C}^{\times}\}$. In our rational model, the centre is given by

$$Z(\mathsf{GU}_{a+b}) := \{(x+iy)I : x, y \in \mathbb{Q}, (x, y) \neq 0\}.$$

We claim that $\mathbb{Q}^2 \backslash \{(0,0)\}$ is not discrete in $\mathbb{A}_f^2 \backslash \{(0,0)\}$. (???)

(SV6) The torus $Z(\mathsf{GU}_{a+b})$ is defined over $\mathbb{Q}$ by the condition of being a diagonal matrix.

Relation to abelian varieties

There is another realization for the $\mathrm{U}_{a,b}$ Shimura variety.

$$\mathrm{U}'_{a,b} := \{g \in \mathrm{GL}_{a+b}(\mathbb{C}) : g^* J_{a,b} g = J_{a,b}\}$$

(using Lan's notation), where

$$J_{a,b} := \begin{bmatrix} & & I_b \\ & S & \\ -I_b & & \end{bmatrix} \in M_{a+b}(\mathbb{C}), \quad S \in M_{a-b}(\mathbb{C})$$

is skew-Hermitian (meaning S is skew-Hermitian) and $(-iS) > 0$.

Lemma

$U_{a,b} \cong U'_{a,b}$.

Proof: Rather arduous matrix computation.

Remark: When $a = b$, $U_{b,b}$ acts on the Hermitian upper half-space

$$\mathcal{H}_{b,b} = \{Z \in M_b(\mathbb{C}) = \mathrm{Herm}_b(\mathbb{C}) \oplus \text{anti-}\mathrm{Herm}_b(\mathbb{C}) : -i(Z - Z^*) > 0\}$$

$J_{a,b}$ defines a skew-Hermitian form on $\mathbb{C}^{a+b}$. The real part $\Re(J_{a,b})$ defines an alternating form on $\mathbb{R}^{2(a+b)}$.

Recall that the map

$$\begin{aligned} \rho\colon U_n(\mathbb{C})' &\to \mathrm{GL}_{2n}(\mathbb{R}) \\ A + Bi &\mapsto \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \end{aligned}$$

is a homomorphism. Additionally, $\rho(Z^*) = \rho(Z)^T$.

If $\Re(J_{a,b})$ is non-degenerate, $\rho(Z) \in \mathrm{Sp}_{2(a+b)}(\Re(J_{a,b}))(\mathbb{R})$.
The conjugacy class of

$$h_0'\colon x + yi \mapsto \begin{bmatrix} x & & y \\ & x - iy & \\ -y & & x \end{bmatrix} \in U_{a,b}'$$

gives rise to Shimura datum $(\mathrm{GU}_{a,b}', \mathrm{GU}_{a,b}' \cdot h_0)$.

$$(\rho \circ h_0')(x+iy) = \left(\begin{array}{ccc|ccc} x & & y & & & \\ & xI_{a-b} & & & -yI_{a-b} & \\ -y & & x & & & \\ \hline & & & x & & y \\ & yI_{a-b} & & & xI_{a-b} & \\ & & & -y & & x \end{array}\right)$$

(Note $(\rho \circ h_0')(i)$ is a different presentation of the standard symplectic form.)

There is a corresponding map of Shimura datum

$$(\mathrm{GU}_{a,b}, D_{a,b}) \mapsto (\mathrm{GSp}_{2a+2b}, X)$$

Example

The points on the Shimura variety $(U_{2,2}, D_{2,2})$ parametrize *some type* of abelian variety. (i.e, some type of polarized Hodge structure.) Too bad the rational points are hard to find :-).

More on $U_{2,2}$

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad \rho(J) = \begin{bmatrix} J & \\ & J \end{bmatrix}, \qquad \rho(A + Bi) = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

By definition, $A^T JA + B^T JB = I$, $A^T JB - B^T JA = 0$. Note

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix}^T \begin{bmatrix} & -J \\ J & \end{bmatrix} \begin{bmatrix} A & B \\ -B & A \end{bmatrix} = \begin{bmatrix} & -J \\ J & \end{bmatrix}$$

There should be an order 4 *automorphism acting on* $H_1(A, \mathbb{Z})$. The fixed order 4 automorphism

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \in \mathrm{Sp}(\rho(J))$$

is centralized by $\rho(U_{2,2})$, but is different than $(\rho \circ h_0')(i)$.

Rational Hodge structure:	$(\mathbb{R}^{2a+2b}, \rho \circ h_x)$
Polarization:	The (fixed) symplectic form $(\rho \circ h_0')(i)$
Extra structure:	Action by an order 4 automorphism.