Dartmouth College

Mathematics 101

Homework 3 (due Wednesday, Oct 13)

1. Let H and K be finite subgroups of a group G. Show that $|HK| = \dfrac{|H|\,|K|}{|H \cap K|}$.
   Note: do not assume that HK is a group. Hint: Consider $(H \cap K)\backslash K$.

2. Let G be a group, and H a normal subgroup. Show that G is solvable if and only if H and G/H are solvable.

3. Show that the following three statements are equivalent. The second is the Feit-Thompson theorem

   (a) A finite non-abelian simple group has even order.

   (b) A simple group of odd order is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ where p is a prime.

   (c) Every group of odd order is solvable.

4. Let G be a group and let G' be the subgroup of G generated by the set $\{xyx^{-1}y^{-1} \mid x, y \in G\}$. G' is called the commutator subgroup of G.

   (a) Show that if H is a subgroup of G, then $H \supseteq G'$ if and only if $H \triangleleft G$ and G/H is abelian. In particular, $G' \triangleleft G$ and G/G' is abelian.

   (b) Show that if $\varphi : G \to H$ is a homomorphism of groups, and H is abelian, then φ factors through G/G', that is there is a map $\varphi_* : G/G' \to H$ with $\varphi = \varphi_* \circ \pi$ with $\pi : G \to G/G'$ the standard quotient map.

5. Let G be a finite group and H a subgroup whose index in G is the smallest prime dividing the order of G. Show that H is normal in G. Hint: Let G act on G/H by left translation.