Dartmouth College

Mathematics 101

Homework 2 (due Wednesday, Oct 4)

1. Let G be a cyclic group of order n, and let $d \in \mathbb{Z}_+$ with $d \mid n$. Show that G has exactly d elements of exponent d.

2. Let G be a group with subgroups, H, K each of finite index in G. Show that $H \cap K$ has finite index in G by establishing the inequality $(G : H \cap K) \leq (G : H)(G : K)$

3. For a group G, the *center* of G, denoted Z_G is defined by:
$Z_G = \{x \in G \mid xg = gx \text{ for all } g \in G\}$. Show that if G/Z_G is cyclic, then G is abelian.

4. Let $Aut(G)$ denote the group of automorphisms of a group G, and $Inn(G)$ the subgroup of inner automorphisms. That is, $Inn(G) = \{\varphi_g \mid g \in G\}$ where $\varphi_g : G \to G$ is defined by $\varphi_g(x) = gxg^{-1}$.

   (a) Show that $Inn(G) \trianglelefteq Aut(G)$.

   (b) Show that $G/Z_G \cong Inn(G)$.

5. Let G be a finite group, and assume N is a normal subgroup of G with $\gcd(|N|, (G : N)) = 1$. Show that N is the only subgroup of G of order $|N|$.