Dartmouth College

Mathematics 101

Homework 6 (due Wednesday, November 8)

1. If R is a ring with identity, an element $e \in R$ is called an *idempotent* if $e^2 = e$. Notice that aside from $e = 0, 1$, all other idempotents are zero divisors. If $\varphi : R \to S$ is a homomorphism between rings with identity, $\varphi(1_R)$ is an idempotent, so to find homomorphisms in which $\varphi(1_R)$ is not the identity, one must look for idempotents in S. Find all *ring* homomorphisms $\varphi : \mathbb{Z}_{120} \to \mathbb{Z}_{42}$; verify they are ring homomorphisms.

2. Let R be a commutative ring with identity. An element $x \in R$ is called *nilpotent*, if $x^n = 0$ for some positive integer n.

   (a) Show that the set of nilpotent elements in R form an ideal, called the *nilradical* of R.

   (b) Show that the sum of a unit and a nilpotent element is a unit in R. *Hint:* As a lemma, show it first with the unit equal to the identity.

3. Let R be a commutative ring with identity. Let $R[x]$ be the polynomial ring in one variable with coefficients in R. Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x]$.

   (a) Show that p is nilpotent in $R[x]$ if and only if a_i is nilpotent in R for all $i \geq 0$.

   (b) Show that p is a unit in $R[x]$ if and only if a_0 is a unit in R and the a_i are nilpotent in R for $i \geq 1$. *Hint:* If $p = a_0 + \cdots + a_n x^n$ and $q = b_0 + \cdots + b_m x^m$ with $pq = 1$, show by induction on r that $a_n^{r+1} b_{m-r} = 0$ to conclude a_n nilpotent for $n \geq 1$.

4. Assume that R is a commutative ring with identity, and let $f(x)$ be a monic polynomial in $R[x]$ of degree $n \geq 1$. Let bar notation denote passage to the quotient ring $R[x]/(f)$.

   (a) Show that every element of $R[x]/(f)$ has a unique representative of the form $\overline{a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}}$ with $a_i \in R$.

   (b) If $f(x) = x^n - a$ for a some nilpotent element of R, show that $\overline{x}$ is nilpotent in $R[x]/(f)$.