Dartmouth College

Mathematics 101

Homework 2 (due Wednesday, Oct 10)

1. Two subgroups H_1 and H_2 of a group G are said to be *commensurable*, denoted $H_1 \sim H_2$, if $H_1 \cap H_2$ has finite index in both H_1 and H_2. We want to establish the commensurability is an equivalence relation (it is obviously reflexive and symmetric).

   (a) Let G be a group with subgroups, H, K each of finite index in G. Show that $H \cap K$ has finite index in G by establishing the inequality $(G : H \cap K) \leq (G : H)(G : K)$.

   (b) Show that commensurability is a transitive relation.

2. Let G_1 and G_2 be groups with normal subgroups H_1 and H_2.

   (a) Show that $H_1 \times H_2 \trianglelefteq G_1 \times G_2$, and that $(G_1 \times G_2)/(H_1 \times H_2) \cong G_1/H_1 \times G_2/H_2$.

   (b) Let p be a prime and n a positive integer. Determine the number of subgroups of $\mathbb{Z}^n$ having index p. Hint: The correspondence theorem and last week's homework should be of help.

3. Let $Aut(G)$ denote the group of automorphisms of a group G, and $Inn(G)$ the subgroup of inner automorphisms. That is, $Inn(G) = \{\varphi_g \mid g \in G\}$ where $\varphi_g : G \to G$ is defined by $\varphi_g(x) = gxg^{-1}$.

   (a) Show that $Inn(G) \trianglelefteq Aut(G)$.

   (b) For a group G, the *center* of G, denoted Z_G is defined by:
   $Z_G = \{x \in G \mid xg = gx \text{ for all } g \in G\}$. Show that $G/Z_G \cong Inn(G)$.

   (c) Show that if G/Z_G is cyclic, then G is abelian.

   (d) Show that if G is a group with cyclic automorphism group, then G is abelian. Hint: In case we haven't covered the structure of cyclic groups yet, subgroups of cyclic groups are cyclic.

4. Let G be a finite group, and assume N is a normal subgroup of G with $\gcd(|N|, (G : N)) = 1$. Show that N is the only subgroup of G of order $|N|$.