Math 101 Fall 2013
Homework #3
Due Wednesday 9 October 2013

1. Recall that a subset S of an R-module is *linearly independent* if given any subset $\{s_1, \cdots, s_m\}$ of distinct elements of S and elements r_i of R such that $r_1 \cdot s_1 + \cdots + r_m \cdot s_m = 0$, then $r_i = 0$ for all i. We call S a *basis* for R if it is linearly independent and generates R (that is, every element of R is a finite linear combination of elements of S). Show that R is free if and only if it has a basis.

2. Give a careful statement of Zorn's Lemma (look it up). Then use Zorn's Lemma to prove that if R is a ring (with identity), then every proper ideal of R is contained in a maximal ideal. In particular, R has a maximal ideal.

3. Recall that the family of subsets of any set are ordered by containment: $A \leq B$ if and only if $A \subset B$. Prove the following assertions that were used without proof in our proof that submodules of free modules are free for modules over at PID.

(a) Let $\mathcal{S} := \{(C, f)\}$ be a nonempty collection of functions $f : C \to A$ where C is a subset of a set B. Order $\mathcal{S}$ by $(C, f) \leq (D, g)$ if $C \subset D$ and $g|_C = f$. Let $\{(C_i, f_i)\}$ be a *totally ordered* subset of $\mathcal{S}$. Define $C = \bigcup C_i$. Show that we get a well-defined function $f : C \to A$ be letting $f(c) = f_i(c)$ if $c \in C_i$.

(b) Let B be a basis for a free module F over R. Let $\{C_i\}$ be a *totally ordered* collection of subsets of B whose union is all of B. Show that $F = \bigcup \langle C_i \rangle$ where, as usual, $\langle C \rangle$ is the submodule of F generated by C. (We don't actually need $\{C_i\}$ to be totally ordered. We just need it to be *cofinal* in that given C_i and C_j there is a C_k containing both of them.)

4. Let V be a finite-dimensional k-vector space and $R : V \to V$ be a linear operator such that $R^2 = \mathrm{id}_V$. Assume the characteristic of k is not 2. Show that V has a basis β such that

$$[R]_\beta^\beta = \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix}$$

where of course I_p is the $p \times p$ identity matrix.

5. Let $f : \mathbf{Z}^n \to \mathbf{Z}^n$ be a $\mathbf{Z}$-module map.

(a) If f is surjective, show that it must also be injective.

(b) If f is injective, it need not be surjective, but show that it must be *almost surjective* in that its cokernel is finite.

(I found the $S^{-1}(\cdot)$ functor helpful.)

6. (Internal coproducts) Let M be an R-module. Suppose there are submodules $\{M_j\}_{j\in J}$ such that

(a) the submodule $\sum_j M_j$ generated by the set $S = \bigcup_j M_j$ is all of M;

(b) and for each j, $M_j \cap \sum_{i\neq j} M_i = \{0\}$.

Then show that M is isomorphic to $\coprod_{j\in J} M_j$ as R-modules.

7. (Primary Decomposition) Let M be a torsion abelian group and let P be the positive primes in $\mathbf{Z}$. For each $p \in P$ and $n \in \mathbf{N}$ let ${}_{p^n}M = \{\, m \in M : p^n \cdot m = 0 \,\}$ be the submodule of M annihilated by p^n. Let $M[p] := \bigcup_{n=1}^{\infty} \bigl({}_{p^n}M\bigr)$. Then $M[p]$ is a submodule of M called the *p-primary component* of M. Show that $M \cong \coprod_{p\in P} M[p]$. (I used question 6 and the observation that if $(a_1, \ldots, a_n) = 1$ — that is, if the integers $a_1, \ldots, a_n$ have no common factor other than 1 — then there are integers b_i such that $b_1a_1 + \cdots + b_na_n = 1$.)