Math 101 Fall 2013
Homework #6
Due Wednesday October 30, 2013

1. Prove Cauchy's Theorem: If p is a prime dividing $|G|$, then G contains an element x of order p. (Since $\langle x \rangle$ is a subgroup of G of order p, we also obtain a *partial* converse to LaGrange's Theorem.)

2. Let H and K be finite subgroups of G.

(a) Prove that

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

(Suggestion: show that the number of distinct left K cosets in HK is equal to the index $H \cap K$ in H.)

(b) Show that if $H \subset N_G(K)$, then HK is a subgroup of G.

(c) Suppose that $H \lhd G$, $K \lhd G$ and $H \cap K = \{1\}$. Show that $HK \cong H \times K$. (Suggestion, if $h \in H$ and $k \in K$, then consider $hkh^{-1}k^{-1}$.)

3. Let F be a finite field and $F^\times$ the multiplicative group of units (a.k.a. the nonzero elements). We want to show that $F^\times$ is cyclic.

(a) Let $G = \mathbf{Z}_{n_1} \times \mathbf{Z}_{n_2} \times \cdots \times \mathbf{Z}_{n_k}$ be a finite abelian group with $n_j \mid n_{j-1}$ for $2 \leq j \leq k$ and $n_j \geq 2$. If we view the operation in G as multiplication with identity 1, how many solutions to $x^{n_1} = 1$ there are in G? (If you write the operation in G additively and use 0 for the identity, this is the same as asking how many solutions to $n_1 \cdot x = 0$ are there?)

(b) Use that fact that in $F[x]$ a polynomial of degree n can have at most n zeros to show that $F^\times$ must be cyclic as claimed.

4. Suppose that $|G| = pqr$ with $p < q < r$ primes. Let P, Q and R be a p-Sylow subgroup, a q-Sylow subgroup and a r-Sylow subgroup, respectively. Show that at least one of P, Q and R is normal in G.

5. Let $|G| = 105$. Suppose that G has a normal 3-Sylow subgroup. Show that $G \cong \mathbf{Z}_{105}$,