Math 101 Fall 2013
Homework #7
Due Friday, November 15, 2013

1. Let R be a unital subring of E. Show that $E \otimes_R R$ is isomorphic to E.

2. Show that $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q}$ and $\mathbf{Q} \otimes_{\mathbf{Q}} \mathbf{Q}$ are isomorphic. (Show both are vector spaces over $\mathbf{Q}$ of dimension one.)

3. Show that as left $\mathbf{R}$-modules, $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$ and $\mathbf{C} \otimes_{\mathbf{C}} \mathbf{C}$ are not isomorphic.

4. Recall that for R-modules, we write $\bigoplus_{i \in I} M_i$ in place of the coproduct $\coprod_{i \in I} M_i$. Then show that tensor products commute with direct sums. That is, show that

$$N \otimes_R \bigoplus_{i \in I} M_i \cong \bigoplus_{i \in I} N \otimes_R M_i,$$

and that an isomorphism is given by $n \otimes (m_i) \mapsto (n \otimes m_i)$. (I suggest using the universal property of the tensor product. What assumptions are you making about R, N and the M_i?)

5. Let A be a finite abelian group of order $p^\alpha m$ with $p \nmid m$. Prove that $\mathbf{Z}_{p^\alpha} \otimes_{\mathbf{Z}} A$ is isomorphic to the p-Sylow subgroup of A.

6. Recall that if S is a multiplicative subset of a commutative ring R and M is an R-module, then we can form the fraction module $S^{-1}M$. Because both share the same universal property, we also observed that $\frac{m}{s} \mapsto \frac{1}{1} \otimes m$ induces an isomorphism of $S^{-1}M$ onto $S^{-1}R \otimes_R M$. Except for part (a), we'll take $R = \mathbf{Z}$ and $S^{-1}R = \mathbf{Q}$ in this problem. But you might want to think about generalizations of parts (b) and (c).

(a) Show that $\frac{1}{s} \otimes m$ is zero in $S^{-1}M \otimes_R M$ if and only if there is a $s' \in S$ such that $s' \cdot m = 0$. ("Use the isomorphism Luke.")

(b) Let A be an abelian group. Show that $\mathbf{Q} \otimes_{\mathbf{Z}} A = \{0\}$ if and only if A is torsion.

(c) Recall that if $\phi : M' \to M$ is an R-module map, then we get a homomorphism $1 \otimes \phi : N \otimes_R M' \to N \otimes_R M$ for any right R-module N characterized by $\phi(n \otimes m') = n \otimes \phi(m')$. Show that if

$$1 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 1$$

is a short exact sequence of abelian groups, then

$$1 \longrightarrow \mathbf{Q} \otimes_{\mathbf{Z}} A \xrightarrow{1 \otimes i} \mathbf{Q} \otimes_{\mathbf{Z}} B \xrightarrow{1 \otimes j} \mathbf{Q} \otimes_{\mathbf{Z}} C \longrightarrow 1$$

is a short exact sequence of vector spaces. (One says that $Q \otimes_{\mathbf{Z}} \_$ is exact. And Luke, don't forget ....)

(d) Is it always true that if M' is a submodule of M then $N \otimes_R M'$ is a submodule of $N \otimes_R M$. (That is, if $\iota : M' \to M$ is the inclusion, is $1 \otimes \iota$ necessarily injective?)

7. In question 4, we observed the tensor products commute with direct sums. Do they commute with direct products? Let P be the set of primes in $\mathbf{Z}$ and consider $M = \prod_{p \in P} \mathbf{Z}_p$ and our old friend $\mathbf{Q} \otimes_{\mathbf{Z}} \_$.

8. If R is a unital subring of E, then we formally have two meanings for $E \otimes_R M$ for an R-module M: we first "extended the scalars from R to E, but we could also consider E as a (E, R)-bimodule and form the (general) tensor product. Explain why these are the same thing.

9. Let $F(S)$ be a free R-module with basis S and let M be an R-module. Show that every element $\mathfrak{t}$ of $F(S) \otimes_R M$ has a *unique* representation in the form

$$\mathfrak{t} = \sum_{s \in S} s \otimes m_s,$$

where only finitely many m_s are nonzero. In particular, if $\mathfrak{t} = 0$, then all the m_s are zero. (If we specialize to the case where S is finite and $F(S) = R^n$, every element of $R^n \otimes_R M$ has a unique representative of the form

$$\sum_{i=1}^{n} e_i \otimes m_i$$

where e_i is the usual basis vector and $m_i \in M$. We also note that it is critical that S be a basis. The set $S = \{s_1, s_2\} = \{(2,0),(0,2)\}$ is $\mathbf{Z}$-linearly independent in $\mathbf{Z}^2$, but $s_1 \otimes 1 + s_2 \otimes 1 = 0$ in $\mathbf{Z}^2 \otimes_{\mathbf{Z}} \mathbf{Z}_2$.)