MATH 101: ALGEBRA I
MIDTERM EXAM

Name ________________________________

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Problem 1. Let $G$ be a group. Indicate if the following statements are true or false. If true, give a proof; if false, give an explicit counterexample.

(a) If $H, H' \trianglelefteq G$ and $G/H \simeq G/H'$, then $H \simeq H'$.

(b) If $H, H' \trianglelefteq G$ and $H \simeq H'$, then $G/H \simeq G/H'$.

(c) If $K, K'$ are groups and $G \times K \simeq G \times K'$, then $K \simeq K'$. 
Problem 2. Let $R$ be a Euclidean domain with norm $N$.

(a) Let

$$m = \min\{N(a) : a \in R, a \neq 0\}.$$

Show that every nonzero $a \in R$ with $N(a) = m$ is a unit in $R$.

(b) Deduce that a nonzero element of norm zero in $R$ is a unit; show by an example that the converse of this statement is false.

(c) Let $F$ be a field and let $R = F[[x]]$. Show that $R$ is Euclidean. What does part (a) tell you about $R^\times$? What are the irreducibles in $R$, up to associates?
Problem 3. Let $F$ be a field and let $V = \text{Mat}_{2\times 3}(F)$ be the $F$-vector space of $2 \times 3$-matrices.

(a) The group $\text{GL}_2(F)$ acts on $V$ by left multiplication. For $M, M' \in V$, the relation $M \sim M'$ if and only if $M' = AM$ for some $A \in \text{GL}_2(F)$ defines an equivalence relation on $V$.

What are the equivalence classes (i.e., the orbits of the action)?

(b) Show that this action $\text{GL}_2(F) \triangleright V$ induces an injective group homomorphism

$$\phi : \text{GL}_2(F) \to \text{Aut}_F(V).$$

(c) Under the isomorphism $\text{Aut}_F(V) \simeq \text{GL}_6(F)$ given by the basis of matrix units, describe $\phi$ explicitly.
Problem 4. For the purposes of this exercise, we say that an isomorphism of $F$-vector spaces is \textit{natural} if it does not depend on a choice of basis.

Let $F$ be a field and let $V, W$ be finite-dimensional vector spaces over $F$. Show that there is a (well-defined) natural isomorphism of $F$-vector spaces

$$
\phi : V^* \otimes_F W \sim \text{Hom}_F(V, W).
$$