MATH 101: ALGEBRA I
WORKSHEET, DAY #3

Fill in the blanks as we finish our first pass on prerequisites of group theory.

1. Subgroups, cosets

Let G be a group. Recall that a subgroup $H \leq G$ is a subset that is a group under the binary operation of G. The subgroup criterion says that $H \subseteq G$ is a subgroup if and only if ____. It shows that if $\phi : G \to G'$ is a group homomorphism, then $\phi(G) \leq G'$ is a subgroup: _____________________________________.

Let $H \leq G$ be a subgroup. Define an equivalence relation on G by $a \sim b$ if and only if $a^{-1}b \in H$. This is an equivalence relation because _____________________________________

____.

The equivalence classes are $aH = \{ah : h \in H\}$ for $a \in G$; note $aH = bH$ if and only if $H = a^{-1}bH$. Let G/H be the set of equivalence classes.

We similarly define equivalence on the other side, with classes Ha for $a \in G$, and write $H\backslash G$ for the set of equivalence classes.

Lagrange's theorem states: ____

____.

It implies Fermat's little theorem: ____

____ because ________________

____.

Date: Wednesday, 14 September 2016.

2. Centralizer, normalizer, stabilizer

Let $A \subseteq G$ be a subset. The centralizer of A in G is $C_A(G) =$ ____________________________, consisting of the elements of G ____. The centralizer $C_A(G) \leq G$ is a subgroup: ____.

The center of G is $Z(G) = C_G(G)$.

Similarly, the normalizer of A in G is $N_A(G) =$ ____________________________. Note that $C_G(A) \leq N_G(A)$. The normalizer $N_A(G) \leq G$ is a subgroup.

Let $G \circlearrowright X$ act on a set X. Let $x \in X$. The stabilizer of $x \in G$ is the set $\mathrm{Stab}_G(x) = G_x = \{g \in G : gx = x\}$. We verify that $G_x \leq G$ is a subgroup. Recall that the kernel of the action is the subgroup ____.

(1) Let $G \circlearrowright X = G$ act on itself by conjugation, so that ____________________________.
Then the centralizer of A in terms of the group action is ____________________________
____.

(2) Let $X = \mathcal{P}(G)$ be the power set of G, consisting of ____________________________________
____________________________. Then $G \circlearrowright X$ acts on X by conjugation via $(g, A) \mapsto gAg^{-1} = \{gag^{-1} : a \in A\}$. Under this action, $N_G(A)$ is ____________________________.

(3) Let the group $N_G(A) \circlearrowright A$ act on A by conjugation. Then $C_G(A) \leq N_G(A)$ is
____________________________.

3. Cyclic groups

A group G is cyclic if ____, and we write ________________________. In this case, we have an isomorphism

____

____

Consequently, any two cyclic groups of the same order are isomorphic.

Lemma. *Let G be a cyclic group, generated by x, and let $a \in \mathbb{Z}$.*

(a) *If x has infinite order, then $G = \langle x^a \rangle$ if and only if* ____________.

(b) *If x has order $n < \infty$, then $G = \langle x^a \rangle$ if and only if* ____________*; in particular, the number of generators of G is* ________________________.

Every subgroup of a cyclic group is cyclic: ____

____

____.

In fact, the subgroups of a finite cyclic group correspond bijectively with ________________________ via ____.

4. Generating subgroups

Let G be a group. Given any nonempty collection $\{H_i\}_{i\in I}$ of subgroups of G, the intersection $\bigcap_{i\in I} H_i \leq G$ is a subgroup.

Let $A \subseteq G$. The subgroup generated by A, written $\langle A \rangle \subseteq G$, is

$$\langle A \rangle = \bigcap_{\substack{A\subseteq H \\ H\leq G}} H.$$

The subgroup generated by A is the smallest subgroup of G containing A in the sense that

____.

A more concrete way of thinking about $\langle A \rangle$ is that it consists of all elements of G that can be written as words in elements of $A \cup A^{-1}$, where $A^{-1} = \{a^{-1} : a \in A\}$: this means

____.

A subgroup $H \leq G$ is finitely generated if

____.

Example. The subgroup of $\mathrm{GL}_2(\mathbb{C})$ generated by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

is ____

____.

Example. The subgroup of $\mathrm{GL}_2(\mathbb{Q})$ generated by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

is infinite even though S, T have orders ____________________ because ____________

____.

5. Kernels, normal subgroups, quotient groups

Let $\phi : G \to G'$ be a group homomorphism. The kernel of ϕ is the subgroup

$$\ker \phi = \underline{\hspace{6cm}} \leq G.$$

Lemma. *ϕ is injective if and only if $\ker \phi = \{1\}$.*

Proof. ____

____. □

Lemma. *Let $K \leq G$ be a subgroup. Then the following conditions are equivalent:*

(i) *For all $a \in G$, we have $aK = Ka$;*

(ii) *For all $a \in G$, we have $aKa^{-1} \subseteq K$; and*

(iii) *For all $a \in G$, we have $aKa^{-1} = K$.*

Proof. Proves itself. □

A subgroup $K \leq G$ is normal if the equivalent conditions (i)–(iii) hold, and we write $K \trianglelefteq G$.

Example. Every subgroup of an abelian group is normal.

Proposition. *If $K \leq G$, then G/K is a group under $(aK)(bK) = abK$ if and only if $K \trianglelefteq G$.*

Proof. ____

____.

□

If $K \trianglelefteq G$, then the group G/K is the quotient group. Kernels of surjective group homomorphisms are the same as normal subgroups in the sense that ________________________________

____.

In general, if $\phi : G \to G'$ is a group homomorphism, then it factors:

Theorem (First isomorphism theorem)**.**

____

There are also three more isomorphism theorems, concerning intersections, quotients, and lattices.

Example. The group $\mathbb{Q}/\mathbb{Z}$ looks like

____.

6. Composition series

A (finite or infinite) group is simple if $\#G > 1$ and ________________________________

________________________.

A composition series for G is a sequence of subgroups

$$1 = N_0 \leq N_1 \leq \cdots \leq N_{k-1} \leq N_k = G$$

such that $N_i \trianglelefteq N_{i+1}$ and N_{i+1}/N_i is a simple group for all $0 \leq i \leq k-1$; we then call the set of groups N_{i+1}/N_i the composition factors.

Example. A composition series for S_3 is:

____

A composition series for D_8 is:

____

A composition series for A_4 is:

____

Example. Let F be a field with $\#F = p$. A composition series for the Heisenberg group $H(F)$ is: ____

Theorem (Jordan–Hölder)**.** *Every nontrivial finite group has a composition series and the composition factors in any composition series are unique (up to reordering).*

To classify all finite groups, we classify all finite simple groups and then try to "put them back together" to get all finite groups.

Theorem. *Every finite simple group is isomorphic to either one in a list of* 18 *infinite families of simple groups or one of* 26 sporadic *simple groups.*

Examples of simple groups include:

- ____
- ____
- ____