MATH 101: ALGEBRA I
WORKSHEET, DAY #6
"LINEAR ALGEBRA EXTRAVANGANZA"

Throughout, let F be a field.

Problem 1. Suppose $\operatorname{char} F \neq 2$. Let V be an F-vector space, and let $\phi, \psi : V \to V$ be projection maps.

(a) Show that $\phi+\psi$ is a projection if and only if $\phi\psi = \psi\phi = 0$ if and only if $\operatorname{img}\phi \subseteq \ker\psi$ and $\operatorname{img}\psi \subseteq \ker\phi$.

(b) If $\phi+\psi$ is a projection, show that $\operatorname{img}(\phi+\psi) = \operatorname{img}(\phi) \oplus \operatorname{img}(\psi)$ and $\ker(\phi+\psi) = \ker(\phi) \cap \ker(\psi)$.

Problem 2. Let V be an F-vector space with $n = \dim_F V < \infty$. Let $A, B \subseteq V$ be F-subspaces with $a = \dim_F A$ and $b = \dim_F B$ and suppose $V = A + B$. Let

$$S = \{f \in \operatorname{End}_F(V) : f(A) \subseteq A,\ f(B) \subseteq B\}.$$

Observe that $S \subseteq \operatorname{End}_F(V)$ is an F-subspace, and then express $\dim_F S$ in terms of n, a, b.

Problem 3. Let V, W be finite-dimensional F-vector spaces, let $X \subseteq W$ be an F-subspace, and let $\phi : V \to W$ be F-linear. Prove that $\dim\phi^{-1}(X)$ is at least $\dim V - \dim W + \dim X$.

Problem 4. Let $A \in \mathrm{M}_n(\mathbb{R})$, and let A^{T} be its matrix transpose. Show that $A^{\mathsf{T}}A$ and A^{T} have the same range.

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