Problem W2.1. Let $\phi: V \to W$ be an $F$-linear map, and let $\phi^*: W^* \to V^*$ be the dual map, defined via pullback. Show that

$$\text{img } \phi^* = \text{ann}(\ker \phi).$$

Solution. We give a direct proof with two containments to show off the key ideas here. For the containment $(\subseteq)$, if $f \in \text{img } \phi^*$, then there exists $g \in W^*$ such that $f = g\phi$; if $x \in \ker \phi$ then $f(x) = g(\phi(x)) = 0$, so $f \in \text{ann}(\ker \phi)$.

Now the other containment $(\supseteq)$. Let $f \in \text{ann}(\ker \phi) \subseteq V^*$; we seek $g \in W^*$ such that $f = g\phi = \phi^* g \in \text{img } \phi^*$. For $y \in \text{img } \phi$, choose $x \in V$ such that $\phi(x) = y$ and define $g(y) = f(x)$; this is well-defined, because if $x' \in V$ also has $\phi(x') = y$, then $x - x' \in \ker \phi$ and thus $f(x) = f(x' + (x - x')) = f(x') + f(x - x') = f(x')$ since $f \in \text{ann}(\ker \phi)$. Therefore $f(x) = g(\phi(x))$ for all $x \in V$ by construction. We extend $g$ to all of $W$ by choosing a complement $X$ to $\ker \phi$ in $W$, so $W = \text{img } \phi \oplus X$, and extending by zero to $X$. Thus $g \in W^*$ and we already showed that $f = g\phi$.

Problem W2.2. Let $V$ be a finite-dimensional vector space over a field $F$, and let $W_1, W_2$ be subspaces.

(a) Prove that $W_1 = W_2$ if and only if $\text{ann}(W_1) = \text{ann}(W_2)$.

(b) Show $\text{ann}(W_1 + W_2) = \text{ann}(W_1) \cap \text{ann}(W_2)$ and $\text{ann}(W_1 \cap W_2) = \text{ann}(W_1) + \text{ann}(W_2)$.

For a heightened sense of self-satisfaction, you could make it clear in your argument where you actually use that $V$ is finite-dimensional. Which of the statements are still true when $V$ is infinite-dimensional?

Solution. For (a), the implication $(\Rightarrow)$ is immediate, so we prove $(\Leftarrow)$. From $\text{ann}(W_1) = \text{ann}(W_2)$ we get $\text{ann}(\text{ann}(W_1)) = \text{ann}(\text{ann}(W_2))$. By daily homework 5.1, we have $\text{ann}(\text{ann}(W)) = \text{ev}(W)$ for a subspace $W$ where $\text{ev}: V \xrightarrow{\sim} V^*$ is the evaluation isomorphism (here is where we use that $V$ is finite-dimensional!), so we have $\text{ev}(W_1) = \text{ev}(W_2)$. Since $\text{ev}$ is an isomorphism, this gives $W_1 = W_2$. The equality 5.1 uses the finite-dimensionality in a key respect (otherwise, we only get a containment), so this argument does not work in the infinite-dimensional case. Here is an argument that does, via contrapositive: start with a basis of $W_1 \cap W_2$ and extend to a basis $\beta_1$ of $W_1$. If $W_1 \neq W_2$, there exists $w_2 \notin W_1$ so $\beta_1 \cap \{w_2\}$ is linearly independent, and we can extend this to a basis $\beta$ for $V$. We define $f(x) = 0$ for $x \in \beta_1$, $f(w_2) = 1$, and extend $f$ by zero on the remaining elements of $\beta$. Then $f \in \text{ann}(W_1)$ but $f \notin \text{ann}(W_2)$, so $\text{ann}(W_1) \neq \text{ann}(W_2)$.

For (b), we begin with the first statement. For $(\supseteq)$, if $f \in \text{ann}(W_1 + W_2)$ then for all $w_1 \in W_1$ we have $f(w_1) = f(w_1 + 0) = 0$ and similarly with $w_2$ so $f \in \text{ann}(W_1) \cap \text{ann}(W_2)$. Conversely, if $f \in \text{ann}(W_1) \cap \text{ann}(W_2)$, then $f(w_1 + w_2) = f(w_1) + f(w_2) = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$, so $f \in \text{ann}(W_1 + W_2)$. This statement uses nothing about finite dimensionality. For the second statement, by (a) it is sufficient to prove the equality after applying annihilators, so we can prove

$$\text{ev}(W_1 \cap W_2) = \text{ann}(\text{ann}(W_1 \cap W_2)) = \text{ann}(\text{ann}(W_1) + \text{ann}(W_2)).$$

The latter, by (a) but plugging in $\text{ann}(W_i)$ for $W_i$, is

$$\text{ann}(\text{ann}(W_1) + \text{ann}(W_2)) = \text{ann}(\text{ann}(W_1)) \cap \text{ann}(\text{ann}(W_2)) = \text{ev}(W_1) \cap \text{ev}(W_2).$$

So we need to assert that the evaluation map preserves intersections, and this is a general property of injective maps (check!). It is also possible to prove the second statement directly, the hard part is to show the inclusion $(\subseteq)$, and for that part here is a sketch: with $f \in \text{ann}(W_1 \cap W_2)$ and divide up $f$ into $f_1$ on $W_1$ and $f_2$ on a complement of $W_1 \cap W_2$ in $W_2$, extending by zero. This alternate proof does not use finite-dimensionality (even though the proof above does!).
**Problem W2.3.** Let $V, W$ be $F$-vector spaces, let $v_1, \ldots, v_n \in V$ be linearly independent, and let $w_1, \ldots, w_n \in W$ be arbitrary. Suppose that
\[
\sum_{i=1}^n v_i \otimes w_i = 0 \in V \otimes_F W.
\]
Show that $w_i = 0$ for all $i = 1, \ldots, n$. Conclude that $v \in V$ and $w \in W$ have $v \otimes w = 0$ if and only if $v = 0$ or $w = 0$.

**Solution.** For a proof using the universal property (in terms of bilinear forms), see Theorem 14.5 in Roman.

Here is a direct argument. We may replace $V$ with $\text{span}(\beta)$ where $\beta = \{v_1, \ldots, v_n\}$ and $W$ similarly, so we may assume that $V, W$ are finite-dimensional. By this reduction, we may also assume that $\beta$ is a basis for $V$. Choose a basis $\gamma$ for $W$. Then under the “crutch” isomorphism, we have
\[
h : V \otimes_F W \sim \rightarrow \text{Mat}_{n \times m}(F)
\]
extended by linearity (multiplying $n \times 1$ by $1 \times m$ to get $n \times m$). Under the coordinate isomorphism $V \rightarrow F^n$ by $v \mapsto [v]_\beta$ we have $[v_i]_\beta = e_i$ the standard basis elements of $F^n$. In the crutch map $h$, multiplying by these standard basis elements just records them in the corresponding row: that is to say, $h(\sum_{i=1}^n v_i \otimes w_i)$ is the $n \times m$-matrix whose $n$ rows are given by $[w_i]_\gamma^T$. We are given that this matrix is zero! So all the rows are zero, so $[w_i]_\gamma = 0$ for all $i$ and so $w_i = 0$ for all $i$.

The second statement is just writing out the special case $n = 1$: if $v = 0$ we are done, otherwise $v$ is linearly independent and applying the first statement gives $w = 0$.

**Problem W2.4.** In class, we showed that the tensor product is characterized by a universal property. Perhaps the simplest situation of a universal property is the following.

Let $X, Y$ be sets. The cartesian product $X \times Y$ has its two projection maps:
\[
\begin{array}{ccc}
X \times Y & \sim \rightarrow & X \\
\pi_X & & \pi_Y \\
\downarrow & & \downarrow \\
Y & & Y
\end{array}
\]

Show that the product $X \times Y$ is universal in this respect: for every set $Z$ and maps
\[
\begin{array}{ccc}
Z & \sim \rightarrow & X \\
f_X & & f_Y \\
\downarrow & & \downarrow \\
Y & & Y
\end{array}
\]
of sets, there exists a unique map $h : Z \rightarrow X \times Y$ such that the diagram
\[
\begin{array}{ccc}
Z & \sim \rightarrow & X \times Y \\
h & & \pi_X \\
\downarrow & & \downarrow \\
X & & Y
\end{array}
\]
commutes.

**Solution.** Let $Z$ be a set with maps as above. We define $h : Z \rightarrow X \times Y$ by $h(z) = (f_X(z), f_Y(z))$. Then $\pi_X(h(z)) = f_X(z)$ and similarly with $Y$, so the diagram commutes. The map $h$ is unique, because if $h'$ fits in the diagram with $h'(z) = (x, y)$, then $f_X(z) = \pi_X(h'(z)) = x$ and similarly $y = f_Y(z)$.

**Problem W2.5.** Let $F$ be a field, let $V$ be a finite-dimensional $F$-vector space, and let $T : V \times V \rightarrow F$ be a nondegenerate symmetric bilinear form. Let $W \subseteq V$ be a subspace.
Define
$$W^\perp = \{ v \in V : T(v, W) = 0 \} = \{ v \in V : T(v, w) = 0 \text{ for all } w \in W \}.$$  

(a) Show that the map
$$V \to V^*, \quad v \mapsto T_v = T(v, -)$$
maps $W^\perp$ isomorphically to $\text{ann}(W)$. 

(b) Deduce that $\dim V = \dim W + \dim W^\perp$. 

(c) Suppose that $T|_{W \times W}$ is nondegenerate (accordingly, we say that $W$ is a nondegenerate subspace under $T$). Show that $V = W \oplus W^\perp$. In this case, we say $W^\perp$ is the orthogonal complement of the nondegenerate subspace $W$. 

(d) Define the orthogonal projection onto $W$ (as a linear operator on $V$). Let $V = \mathbb{R}^3$ have the standard inner product and let 
$$W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0 \}.$$ 
Compute the matrix of the orthogonal projection onto $W$ with respect to the standard basis.

Solution. For part (a), the map $V \to V^*$ is linear, injective by definition that the pairing is nondegenerate, so it is an isomorphism. So it is enough to show that the image of $W^\perp$ is $\text{ann}(W)$. But $v \in W^\perp$ if and only if $T(v, w) = T_v(w) = 0$ for all $w \in W$ if and only if $T_v \in \text{ann}(W)$.

For part (b), we showed in class that $\dim \text{ann}(W) = V - \dim W$, so by (a) $\dim W^\perp = V - \dim W$ as well.

For part (c), we always have $W + W^\perp \subseteq V$. The sum is direct because if $w \in W \cap W^\perp$ then $T(w, w') = 0$ for all $w' \in W$, so since $T|_{W \times W}$ is nondegenerate, we must have $w = 0$. The containment $W \oplus W^\perp = V$ is an equality by dimensions: $\dim (W \oplus W^\perp) = \dim W + \dim W^\perp = \dim V$ by (b).

For (d), the answer is 
$$\frac{1}{2} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$ One efficient way to get this is to take the basis $(1, -1, 0), (0, 1, -1)$ for $W$, extend to a basis $\beta$ of $V$ by adding $(1, 1, 1) \in W^\perp$, so $A = [\phi]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; to get the matrix in the standard basis, conjugate by the change of basis matrix
$$P = [\text{id}]_\beta^\beta = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$
to get
$$[\phi]_\beta = P A P^{-1} = [\text{id}]_\beta^\beta [\phi]_\beta [\text{id}]_\beta^\beta.$$

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