Problem W2.1. Let \( \phi : V \to W \) be an \( F \)-linear map, and let \( \phi^* : W^* \to V^* \) be the dual map, defined via pullback. Show that

\[
\text{img} \phi^* = \text{ann} (\ker \phi).
\]

Problem W2.2. Let \( V \) be a finite-dimensional vector space over a field \( F \), and let \( W_1, W_2 \) be subspaces.

(a) Prove that \( W_1 = W_2 \) if and only if \( \text{ann}(W_1) = \text{ann}(W_2) \).

(b) Show \( \text{ann}(W_1 + W_2) = \text{ann}(W_1) \cap \text{ann}(W_2) \) and \( \text{ann}(W_1 \cap W_2) = \text{ann}(W_1) + \text{ann}(W_2) \).

For a heightened sense of self-satisfaction, you could make it clear in your argument where you actually use that \( V \) is finite-dimensional. Which of the statements are still true when \( V \) is infinite-dimensional?

Problem W2.3. Let \( V, W \) be \( F \)-vector spaces, let \( v_1, \ldots, v_n \in V \) be linearly independent, and let \( w_1, \ldots, w_n \in W \) be arbitrary. Suppose that

\[
\sum_{i=1}^{n} v_i \otimes w_i = 0 \in V \otimes_F W.
\]

Show that \( w_i = 0 \) for all \( i = 1, \ldots, n \). Conclude that \( v \in V \) and \( w \in W \) have \( v \otimes w = 0 \) if and only if \( v = 0 \) or \( w = 0 \).

Problem W2.4. In class, we showed that the tensor product is characterized by a universal property. Perhaps the simplest situation of a universal property is the following.

Let \( X, Y \) be sets. The cartesian product \( X \times Y \) has its two projection maps:

\[
\begin{array}{c}
X \\
\downarrow \pi_X \\
X \times Y \\
\downarrow \pi_Y \\
Y
\end{array}
\]

Show that the product \( X \times Y \) is \textit{universal} in this respect: for every set \( Z \) and maps

\[
\begin{array}{c}
Z \\
\downarrow f_Z \\
X \\
\downarrow f_X \\
Y
\end{array}
\]

Date: Assigned Friday, 22 September 2017; due Monday, 2 October 2017.
of sets, there exists a unique map \( h : Z \rightarrow X \times Y \) such that the diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{h} & X \times Y \\
\downarrow f_X & & \downarrow \pi_X \\
X & \xrightarrow{\pi_Y} & Y
\end{array}
\]

commutes.

**Problem W2.5.** Let \( F \) be a field, let \( V \) be a finite-dimensional \( F \)-vector space, and let \( T : V \times V \rightarrow F \) be a nondegenerate symmetric bilinear form. Let \( W \subseteq V \) be a subspace.

Define

\[
W^\perp = \{ v \in V : T(v, W) = 0 \} = \{ v \in V : T(v, w) = 0 \text{ for all } w \in W \}.
\]

(a) Show that the map

\[
V \rightarrow V^* \\
v \mapsto T_v = T(v, -)
\]

maps \( W^\perp \) isomorphically to \( \text{ann}(W) \).

(b) Deduce that \( \dim V = \dim W + \dim W^\perp \).

(c) Suppose that \( T|_{W \times W} \) is nondegenerate (accordingly, we say that \( W \) is a nondegenerate subspace under \( T \)). Show that \( V = W \oplus W^\perp \). In this case, we say \( W^\perp \) is the orthogonal complement of the nondegenerate subspace \( W \).

(d) Define the orthogonal projection onto \( W \) (as a linear operator on \( V \)). Let \( V = \mathbb{R}^3 \) have the standard inner product and let

\[
W = \{ (x, y, z) \in \mathbb{R}^3 : x + y + z = 0 \}.
\]

Compute the matrix of the orthogonal projection onto \( W \) with respect to the standard basis.