**MATH 101: GRADUATE LINEAR ALGEBRA**

**WEEKLY HOMEWORK #3 SOLUTIONS**

**Problem W3.1.** Let $V, W$ be finite-dimensional inner product spaces.

(a) Let $\phi: V \rightarrow V$ be a self-adjoint linear operator. Recall that $\psi: V \rightarrow V$ is positive semidefinite if $\langle \psi(x), x \rangle \geq 0$ whenever $x \neq 0$. Show that $\phi$ is positive semidefinite if and only if all the eigenvalues of $\phi$ are nonnegative.

(b) Now let $\phi: V \rightarrow W$ be linear. Show that $\phi^* \phi$ and $\phi \phi^*$ are positive semidefinite.

(c) Show that $\text{rk}(\phi^* \phi) = \text{rk}(\phi \phi^*) = \text{rk}(\phi)$.

**Solution.** For (a), since $V$ is self-adjoint, there is an orthonormal basis $\beta = \{v_i\}_i$ of $V$ of eigenvectors for $V$ with corresponding eigenvalues $\lambda_i$. If $\phi$ is positive semidefinite, then

$$\langle \phi(v_i), v_i \rangle = \lambda_i \|v_i\|^2 \geq 0$$

so $\lambda_i \geq 0$ for all $i$; conversely, if $\lambda_i \geq 0$ for all $i$, then for all $x = \sum_i a_i v_i \in V$ we have

$$\langle \phi(x), x \rangle = \sum_{i,j} a_i \overline{a_j} \lambda_i \langle v_i, v_j \rangle = \sum_i \lambda_i |a_i|^2 \geq 0$$

for all $i$, the latter because $\{v_i\}_i$ is orthonormal so $\langle v_i, v_j \rangle = \delta_{ij}$.

For (b), we noted in class that $\phi^* \phi$ and $\phi \phi^*$ are self-adjoint. We have

$$\langle \phi^* \phi(x), x \rangle = \langle \phi(x), \phi(x) \rangle \geq 0$$

for all $x \in V$, so $\phi^* \phi$ is positive semidefinite; a similar argument works for $\phi \phi^*$.

For (c), we keep the notation from (b) that $\phi: V \rightarrow W$ is linear. We already showed in daily homework 8.1 that $\ker \phi^* \phi = \ker \phi$; rank-nullity applies, so $\text{rk}(\phi^* \phi) = \text{rk}(\phi)$. We also showed in class that $\text{rk}(\phi) = \text{rk}(\phi^*)$ (strictly speaking, only for the dual not the adjoint, but these are related by a change-of-basis); so applying the previous with $\phi^*$ in place of $\phi$ gives the other equality.

**Problem W3.2.** Let $V = \mathbb{R}^n$ be the standard inner product space. Let

$$S = \{x \in V : \|x\|^2 = 1\}$$

be the $(n-1)$-dimensional sphere in $V$.

(a) Suppose that $x, y \in S$ have $\langle x, y \rangle = 0$. Show that $\cos(t)x + \sin(t)y$ lies on $S$ for all $t \in \mathbb{R}$.

(b) Let $\phi: V \rightarrow V$ be a self-adjoint linear map. By vector calculus, the function $x \mapsto \langle x, \phi(x) \rangle$ achieves a maximum at some point $p \in S$: briefly explain why. Let $y \in S$ satisfy $\langle p, y \rangle = 0$. Consider the function

$$f(t) = \langle \cos(t)p + \sin(t)y, \phi(\cos(t)p + \sin(t)y) \rangle.$$ 

Show that $\langle p, \phi(y) \rangle = 0$.

(c) Let $W = \text{span}\{p\}$. Show that $W^\perp$ is $\phi$-invariant and then conclude that $W$ is $\phi$-invariant. Conclude that $p$ is an eigenvector!

(d) Parlay the argument of (c) into an inductive proof that $V$ has an orthonormal basis of vectors that are eigenvectors for $\phi$.

[Note: This argument inductively gives a different “physical” or “geometric” proof that $\phi$ has an orthonormal basis of eigenvectors: we find an eigenvector by maximizing $\phi$ on the sphere!]

**Solution.** For (a), we see that since $x$ and $y$ are perpendicular, so are their scalar multiples, hence by Pythagoras

$$\| \cos(t)x + \sin(t)y \|^2 = |\cos(t)|^2 \|x\|^2 + |\sin(t)|^2 \|y\|^2 = |\cos(t)|^2 + |\sin(t)|^2 = 1.$$
For (b), the maximum exists by the extreme value theorem, because \( f \) is continuous and \( S \) is closed and bounded. We have

\[
f(t) = \langle \cos(t)p + \sin(t)y, \phi(\cos(t)p + \sin(t)y) \rangle \\
= \cos^2(t) \langle p, \phi(p) \rangle + \cos(t) \sin(t) \langle p, \phi(y) \rangle + \langle y, \phi(p) \rangle + \sin^2(t) \langle y, \phi(y) \rangle
\]

so that

\[
f'(t) = -2 \cos(t) \sin(t) \langle p, \phi(p) \rangle + \\
(\cos^2(t) - \sin^2(t)) \langle p, \phi(y) \rangle + 2 \cos(t) \sin(t) \langle y, \phi(y) \rangle
\]

and

\[
f'(0) = \langle p, \phi(y) \rangle + \langle y, \phi(p) \rangle = 0
\]

because \( p \) is a maximum obtained at \( t = 0 \) and so is a critical point of the differentiable function \( f \). If \( \phi = \phi^* \), then

\[
\langle y, \phi(p) \rangle = \langle \phi^*(y), p \rangle = \langle \phi(y), p \rangle = \langle p, \phi(y) \rangle
\]

since our base field is real, and therefore \( \langle p, \phi(y) \rangle = 0 \).

For part (c), if \( x \in W^\perp \) then \( \langle p, x \rangle = 0 \) which by (b) implies \( \langle p, \phi(x) \rangle = 0 \) so \( \phi(x) \in W^\perp \) as claimed. Therefore \( W^\perp \) is \( \phi \)-invariant, and since \( V = W \oplus W^\perp \), \( W \) is also \( \phi \)-invariant (explicitly, if \( z \in W \) then \( \langle y, \phi(z) \rangle = 0 \) for all \( y \in W^\perp \) so \( \phi(z) \in (W^\perp)^\perp = W \). Since \( W \) is one-dimensional, we have \( \phi(p) = cp \) for some \( c \in \mathbb{R} \), which is to say, \( p \) is an eigenvector!

To prove (d), we note that the result holds trivially for the case \( n = 1 \); for general \( n \), choose \( p \in S \) and by (c) we have \( p \) is an eigenvector with eigenvalue \( c \). Restricting \( \phi \) to \( W^\perp \), which gives an operator since \( W^\perp \) is \( \phi \)-invariant, we have an \( n-1 \)-dimensional space so by induction, there exists a basis for \( W^\perp \) which consists of eigenvectors for \( \phi \). Since \( V = W \oplus W^\perp \), the union of this basis with \( p \) gives such a basis for \( V \); in this basis, \( \phi \) is diagonal.

**Problem W3.3.** In each part, let \( \phi: V \to V \) be the projection on the subspace \( W_1 \) along the subspace \( W_2 \), where \( V = W_1 \oplus W_2 \).

(a) Show that \( \phi \) is an orthogonal projection (i.e., \( W_2 = W_1^\perp \)) if and only if \( \|\phi(x)\| \leq \|x\| \) for all \( x \in V \).

*Hint: Let \( w_1 \in W_1 \) and \( w_2 \in W_2 \) be nonzero and \( c \in \mathbb{R} \), and let \( w = cw_1 + w_2 \). Show that
\[
2c \text{Re} \langle w_1, w_2 \rangle + \|w_2\|^2 \geq 0.
\]

If \( \langle w_1, w_2 \rangle \neq 0 \), derive a contradiction by a choice of \( c \); conclude that \( \langle w_1, w_2 \rangle = 0 \) for all \( w_1, w_2 \).

(b) What can you conclude if \( \phi \) is unitary? [So don’t confuse a projection that is orthogonal with an orthogonal projection!]

(c) Suppose that \( \phi \) is normal (over \( \mathbb{C} \)). Prove that \( \phi \) is an orthogonal projection.

**Solution.** For (a), we first prove \( \Rightarrow \). Let \( y \in V \) and write \( W = W_1 \). Since \( V = W \oplus W^\perp \) by previous homework, we can write uniquely \( y = u + z \), where \( u \in W \) and \( z \in W^\perp \), so then \( \phi(y) = u \). We compute that

\[
\|\phi(y)\|^2 = \langle \phi(y), \phi(y) \rangle = \langle u, u \rangle
\]

and

\[
\|y\|^2 = \langle y, y \rangle = \langle u + z, u + z \rangle = \langle u, u \rangle + \langle u, z \rangle + \langle z, u \rangle + \langle z, z \rangle.
\]

Since \( u \in W \) and \( z \in W^\perp \), \( \langle u, z \rangle = \langle z, u \rangle = 0 \). Therefore

\[
\|\phi(y)\|^2 = \|u\|^2 \leq \|u\|^2 + \|z\|^2 = \|y\|^2,
\]

the latter by the Pythagorean theorem. OK, we have used that twice now, so here’s the proof: in general, we have

\[
\|x \pm y\|^2 = \langle x \pm y, x \pm y \rangle \\
= \langle x, x \rangle \pm \langle x, y \rangle \pm \langle y, x \rangle + \langle y, y \rangle \\
= \|x\|^2 \pm \langle x, y \rangle + \langle y, x \rangle + \|y\|^2 \\
= \|x\|^2 \pm 2 \text{Re} \langle x, y \rangle + \|y\|^2
\]
so \( x, y \) are orthogonal if and only if \( \langle x, y \rangle = 0 \) if and only if
\[
\|x + y\|^2 = \|x\|^2 + \|y\|^2.
\]

Continuing with (a), we prove \((=)\). If \( \phi \) is the zero or identity operator, then the result is trivial. So suppose \( \phi \) is projection on \( W_1 \) along \( W_2 \), and let \( w_1 \in W_1 \) and \( w_2 \in W_2 \) be nonzero vectors, and let
\[
w = cw_1 + w_2 \quad \text{and} \quad c \in \mathbb{R}.
\]
Then
\[
\|cw_1\|^2 = \|\phi(w)\|^2 \leq \|w\|^2 = \langle w, w \rangle = \langle cw_1 + w_2, cw_1 + w_2 \rangle = \|cw_1\|^2 + 2c \Re \langle w_1, w_2 \rangle + \|w_2\|^2
\]
so
\[
0 \leq 2c \Re \langle w_1, w_2 \rangle + \|w_2\|^2.
\]
Assume for purposes of contradiction that \( \langle w_1, w_2 \rangle \neq 0 \). If \( \Re \langle w_1, w_2 \rangle = 0 \) then we are over \( F = \mathbb{C} \), replace \( w_2 \) by \( iw_2 \) so that \( \Re \langle w_1, iw_2 \rangle = -\Im \langle w_1, w_2 \rangle \neq 0 \). Then choose \( c < -\|w_2\|^2/\Re \langle w_1, w_2 \rangle \), and we have a contradiction. So \( \langle w_1, w_2 \rangle = 0 \) for all \( w_1, w_2 \), so \( \phi \) is an orthogonal projection.

For (b), if \( \phi \) is unitary, then \( \phi \) is invariant! So then \( \ker \phi = \{0\} = W_2 \) so \( W_1 = V \) and \( \phi \) is the identity.

Finally, part (c). Since \( \phi \) is a projection, it has only eigenvalues 0 and 1: if \( \phi \) is the projection on \( W_1 \) along \( W_2 \), then \( \phi(w_1) = w_1 \) for all \( w_1 \in W_1 \) and \( \phi(w_2) = 0 \) for all \( w_2 \in W_2 \). Since \( \phi \) is normal, there exists an orthonormal basis of eigenvectors of \( \phi \), so in particular there exists an orthonormal basis for \( W_1, W_2 \), respectively. But \( W_1 \) is orthogonal to \( W_2 \), so \( \phi \) is an orthogonal projection.

**Problem W3.4.** Let \( \phi, \psi : V \to V \) be normal operators on a finite-dimensional complex inner product space \( V \). Suppose that \( \phi \psi = \psi \phi \). Prove that there exists an orthonormal basis for \( V \) consisting of (simultaneous) eigenvectors for \( \phi \) and \( \psi \).

**Solution.** It is probably cleanest to work by induction on \( n = \dim V \). (Strictly speaking, we just need a notational device to keep track of eigenvalues, so this will generalize to a certain extent.)

Let \( \lambda \) be an eigenvalue of \( \phi \) (exists since \( \phi \) is normal) with eigenspace
\[
W = E_\lambda = \{ x \in V : \phi(x) = \lambda x \}
\]
for \( \phi \). Then visibly \( \phi(W) \subseteq W \): recall we say that \( W \) is \( \phi \)-invariant. But the commutativity condition also implies that \( W \) is \( \psi \)-invariant: if \( x \in W \) then
\[
\phi(\psi(x)) = \psi(\phi(x)) = \psi(\lambda x) = \lambda \psi(x)
\]
so we can consider \( \psi|_W : W \to W \) the restriction. Then \( \psi|_W \) is still normal, so there exists an orthonormal basis \( \beta_W = \{ w_1, \ldots, w_k \} \) of eigenvectors of \( \psi|_W \) for \( W \) (with some eigenvalues we do not know). But then of course \( \beta_W \) is also an orthonormal basis of eigenvectors for \( \phi|_W \), because they came from the eigenspace \( W = E_\lambda \). So we have a simultaneous basis of eigenvectors.

If \( W = V \), we are done; otherwise, we write \( V = W \oplus W^\perp \). As we have checked before, since \( W \) is \( \phi \)-invariant, so too is \( W^\perp \) and the same with \( \psi \), so by induction we find a simultaneous basis of eigenvectors for \( W^\perp \), and we add the eigenvectors in the previous paragraph to obtain a simultaneous basis of eigenvectors for \( V \).