Math 102
Foundations of Smooth Manifolds
Fall 2011
Assignment 3
Due October 19, 2011

1. Boothby IV.2.10
2. Boothby IV.3.6
3. Boothby IV.3.8
4. Boothby IV.3.9
5. Let $F : \mathbb{R}^3 \to \mathbb{R}^4$ be given by $F(x, y, z) = (x^2 - y^2, xy, xz, yz)$ and let $S^2 \subset \mathbb{R}^3$ be the unit sphere centered at the origin of $\mathbb{R}^3$. Now observe that for any $p \in S^2$ we have $F(p) = F(-p)$, so we obtain an induced map $\widetilde{F} : \mathbb{R}P^2 \to \mathbb{R}^4$ given by $\widetilde{F}([p]) = F(p)$. Show that
   (a) $\widetilde{F}$ is an immersion.
   (b) $\widetilde{F}$ is injective.
   (c) $\widetilde{F}$ is an imbedding.
6. Consider $G = \mathrm{GL}_2(\mathbb{R})$ with the usual C^∞ structure generated by the atlas $\mathcal{A} = \{(\mathrm{GL}_2(\mathbb{R}), \phi)\}$, where $\phi : \mathrm{GL}_2(\mathbb{R}) \to W \subset \mathbb{R}^4$ given by

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \mapsto (x_1, x_2, x_3, x_4),$$

   and $W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1x_4 - x_2x_3 \neq 0\}$. Then $G \times G$ has the C^∞-structure generated by the atlas $\{(G \times G, \phi \times \phi)\}$. Let $m : G \times G \to G$ be the multiplication map, $i : G \to G$ the inversion map and I denote the identity matrix.
   (a) Compute the matrix of $m_* : T_{(I,I)}G \times G \to T_I G$ relative to the coordinate frames induced by the charts above. Conclude that $m_* : T_{(I,I)}G \times G \equiv T_I G \times T_I G \to T_I G$ is just addition.
   (b) Compute the matrix of $i_* : T_I G \to T_I G$ relative to the coordinate frames induced by the charts above. Conclude that $i_* : T_I G \to T_I G$ is given by $X \mapsto -X$?
7. (Lee 1-5) Let $N = (0, 0, \ldots, 0, 1)$ be the "north pole" in $S^n \subset \mathbb{R}^{n+1}$, and let $S = -N$ be the "south pole." Define *stereographic projection* $\sigma : S^n \backslash \{N\} \to \mathbb{R}^n$ by

$$\sigma(x_1, \ldots, x_{n+1}) = \frac{(x_1, \ldots, x_n)}{1 - x_{n+1}}.$$

   Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in S^n \backslash \{S\}$.
   (a) For any $x \in S^n \backslash \{N\}$, show that $\sigma(x)$ is the point where the line through N and x intersects the linear subspace where $x_{n+1} = 0$ (identified) with $\mathbb{R}^n$ in the obvious way). Similarly, show that $\tilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace.
   (b) Show that σ is bijective and

$$\sigma^{-1}(u_1, \ldots, u_n) = \frac{(2u_1, \ldots, 2u_n, \|u\|^2 - 1)}{\|u\|^2 + 1}.$$

(c) Compute the transition map $\tilde{\sigma}\circ\sigma^{-1}$ and verify that the atlas $\mathcal{A}_{\text{stereo}} = \{(S^n\backslash\{N\}, \sigma), (S^n\backslash\{S\}, \tilde{\sigma})\}$ defines a smooth structure on S^n.

(d) Is the smooth structure generated by $\mathcal{A}_{\text{stereo}}$ the same as that generated by $\mathcal{A}_{\text{hem}} = \{(U_i^{\pm}, \phi_i^{\pm})\}_{i=1}^{n+1}$, the atlas constructed in class?

8. Let M be a smooth n-manifold with differentiable structure $\mathcal{U}$. We say that M is *orientable* (with respect to $\mathcal{U}$) if there is an atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha\in J} \subset \mathcal{U}$ so that for each $\alpha, \beta \in J$ such that $U_\alpha \cap U_\beta \neq \emptyset$ the differential

$$(\phi_\beta \circ \phi_\alpha^{-1})_* : T_{\phi_\alpha(p)}\phi_\alpha(U_\alpha \cap U_\beta) \to T_{\phi_\beta(p)}\phi_\beta(U_\alpha \cap U_\beta)$$

has positive determinant for all $p \in U_\alpha \cap U_\beta$. Otherwise, we say M is *non-orientable*.

(a) Show that S^n with the usual differentiable structure is orientable.

(b) Show that for any smooth manifold M, its tangent bundle TM equipped with the usual differentiable structure is orientable.