Math 102
Foundations of Smooth Manifolds
Fall 2011
Assignment 5
Due November 21, 2011

1. (Lee 19-7) Let $U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_i > 0\}$ now let $X = x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}$ and $Y = x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3}$. The vector fields X and Y determine a smooth *involutive* 2-plane distribution Δ on U. Find an explicit flat coordinate system for this distribution in the neighborhood of $(1, 1, 1)$. (**Hint:** Use the method outlined in class.)

2. Boothby IV.7.2

3. Boothby IV.7.4

4. Boothby IV.7.5

5. Boothby IV.8.2

6. (**10 Points**) The point of the following exercises is to prove the following theorem:

   Theorem 0.1. *Let G be a connected abelian Lie group, then G is isomorphic as a Lie group to $T^k \times R^m$.*

   (a) Show that a non-trivial discrete subgroup Γ of a finite dimensional real vector space V is generated by linearly independent vectors $\gamma_1, \gamma_2 \ldots, \gamma_k \in V$. (**Hint:** Use induction on the dimension of V. The case where $\dim V = 1$ is fairly clear (?). Now assume it is true for n. Now consider V of dimension $n + 1$ and put an inner product $\langle \cdot, \cdot \rangle$ and let $\gamma_1 \neq 0 \in \Gamma$ have the smallest positive norm and let W be the orthogonal complement of $\mathbb{R} \cdot \gamma_1$. Now let $\pi : V = \mathbb{R} \cdot \gamma_1 \oplus W \to W$ be the projection. Then $\pi(\Gamma)$ is a subgroup of W, that does not contain a non-zero element with norm less than $\frac{|\gamma_1|}{2}$ (why?). Then $\pi(\Gamma)$ is a discrete subgroup of W and by induction is generated by linearly independent vectors $\hat{\gamma}_2, \ldots, \hat{\gamma}_k \in W$ where $k \leq n + 1$. So $\pi : \Gamma \to \langle \hat{\gamma}_2, \ldots, \hat{\gamma}_k \rangle$ has kernel $\langle \gamma_1 \rangle$ and we obtain a short exact sequence:

   $$0 \to \langle \gamma_1 \rangle \to \Gamma \xrightarrow{\pi} \langle \hat{\gamma}_2, \ldots, \hat{\gamma}_k \rangle \to 0.$$

   This is a split exact sequence (why?), which allows you to conclude what about Γ?)

   (b) Let G be a connected Lie group. Then for any U neighborhood of the identity element, we have $G = \langle U \rangle$; that is, G is generated by U. (**Hint:** Consider V a neighborhood of e contained in U such that $V = V^{-1} \equiv \{x^{-1} : x \in V\}$ (why should this exist?) and let $H = \cup_{n=1}^{\infty} V^n$, where $V^n \equiv \{x_1 \cdot x_n : x_i \in V\}$. Then H is a non-trivial subgroup of $\langle U \rangle$. Show that H is both open and closed in G. Hence?)

   (c) Show that if G is a connected abelian Lie group, then $\exp : T_eG \to G$ is a surjective Lie group homomorphism. (**Hint:** the naturality of the exponential map and the fact that in this case multiplication is a Lie group homomorphism, might prove to be useful.) (Please note than in general exp is *not* a homomorphism nor is it surjective, even when G is connected.)

   (d) Show that if G is a connected abelian Lie group, then the kernel of $\exp : T_eG \to G$ is a discrete subgroup of T_eG.

   (e) Prove the theorem.