Homework for Math 103
Assignment One
Due Wednesday, October 1, 2008

1. Show that the countable union of sets of content zero has content zero. (Use the fact that a countable union of countable sets is countable.)

2. Show that any monotonic function on $[a, b]$ is Riemann integrable on $[a, b]$. (While it is possible to argue that any monotonic function can have at most countably many discontinuities and then apply our general criteria (Theorem 8 from lecture), I want you to prove this using the definition and whatever we have proved so far in lecture. For example, you could take a regular partition and try to apply Theorem 5.)

3. Prove Theorem 10 under the additional assumption that f is continuous. (Recall that a continuous function on a closed bounded interval is uniformly continuous and have a look at the proof that continuous functions must be Riemann integrable.)

In questions 4 and 5, you are encouraged to use the library to check definitions, find examples or even find the argument itself. (Of course, you should cite any sources you end up using.)

4. Let $\{ f_n \}$ be a sequence of real-valued functions on $[0, 1]$. State carefully what it means for $\{ f_n \}$ to converge pointwise to a function f, and what it means for $\{ f_n \}$ to converge uniformly to f on $[0, 1]$. Give a example of a sequence of continuous functions which converges pointwise to the zero function, but which does not converge uniformly. Justitfy your assertions.

5. Suppose that $\{ f_n \}$ is a sequence of Riemann integrable functions on $[a, b]$. If $\{ f_n \}$ converges uniformly to f on $[a, b]$, must f be Riemann integrable? If f is Riemann integrable, then must it be true that

$$\lim_{n\to\infty} \int_a^b f_n \to \int_a^b f?$$

What happens if uniform convergence is replaced by pointwise convergence in the above?