Homework for Math 103
Assignment Two
Due Monday, October 20, 2008

1. Recall from calculus that if $\{ a_n \}$ is a sequence of nonnegative real numbers, then $\sum_{n=1}^{\infty} = \sup_n s_n$, where $s_n := a_1 + \cdots + a_n$. (Note that the value ∞ is allowed.)

(a) Show that $\sum_{n=1}^{\infty} a_n = \sup\{ \sum_{k \in F} a_k : F$ is a finte subset of $\mathbf{Z}^+ = 1, 2, 3, \dots \}$.

(The point of this part of the problem is that if I is any set — countable or not — and if $a_i \geq 0$ for all $i \in I$, then we can define

$$\sum_{i \in I} a_i := \sup\Big\{ \sum_{i \in F} a_i : F \text{ is a finite subset of } I \Big\},$$

and our new definition coincides with the usual one from calculus when both make sense.)

(b) Let X be a set and $f : X \to [0, \infty)$ a function. For each $E \subset X$, define

$$\nu(E) := \sum_{x \in E} f(x).$$

Show that ν is a measure on $\big(X, \mathcal{P}(X)\big)$.

(Some special cases are of note: if $f(x) = 1$ for all $x \in X$, then ν is called *counting measure* on X. If $f(x) = 0$ for all $x \neq x_0$ and $f(x_0) = 1$, then ν is called the *Dirac delta measure at* x_0. If $\sum_{x \in X} f(x) = 1$, then f is a discrete probability distribution on X, and $\nu(E)$ is the probability of the event E for this distribution.)

(c) Let X, f and ν be as in part (b). Show that if $\nu(E) < \infty$, then $\{ x \in E : f(x) > 0 \}$ is countable. (Hint: if $\{ x \in E : f(x) > 0 \}$ is uncountable, then at least one of the sets $\{ x \in E : f(x) > \frac{1}{m} \}$ must be infinite.)

(The result in part (c) shows that discrete probability distributions "live on" countable sample spaces.)

2. Suppose that $\mathscr{M}$ is an *infinite* sigma algebra in a set X. Show that card$(\mathscr{M}) \geq \mathfrak{c}$, were $\mathfrak{c}$ is the cardinality of the continuium (see §0.3 in the text). (Hint: if $\mathscr{M}$ is countable, then each $x \in X$ is contained in a smallest measurable set $A(x) \in \mathscr{M}$.)

3. Show that every σ-finite measure is ***semifinite***. (This is problem #13 in §1.3 of the text.)

4. Suppose that μ is a ***semifinite*** measure and that $\mu(E) = \infty$. Show that for any $C > 0$, there is a $F \subset E$ such that $C < \mu(F) < \infty$. (This is problem #14 in §1.3 of the text.)

5. Prove Proposition 1.20 in the text. (This is Littlewood's First Principle. Recall that every open set in $\mathbf{R}$ is a disjoint union of open intervals and use Theorem 27 from lecture.)

6. Let $F : \mathbf{R} \to \mathbf{R}$ be given by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } 0 < x < 1, \\ 2 & \text{if } 1 \leq x < 2, \text{ and} \\ 3 & \text{if } 2 \leq x. \end{cases}$$

Let μ be the Lebesgue-Stieltjes measure associated to F. Find the measure of the following sets: $[0,1]$, $(0,1)$, $(1,2)$ and $[1,2]$ as well as the singletons $\{0\}$ and $\{1\}$. Of course, I expect you to justify your assertions.