Theorem 1 (Folland Theorem 2.28)**.** *Suppose that f is a bounded real-valued function on $[a, b]$.*

1. *If f is Riemann integrable, then f is Lebesgue measurable (and therefore integrable). Furthermore*

$$\mathcal{R}\int_a^b f = \int_{[a,b]} f(x)\,dm(x). \tag{1}$$

(Henceforth, we will dispense with the notations in (1) *and write simply $\int_a^b f(x)dx$.)*

2. *Also, f is Riemann integrable if and only if the set of discontinuities of f has measure zero.*

Proof. Let $\mathcal{P} = \{\, a = t_0 < t_1 < \cdots < t_n = b \,\}$ be a partition of $[a, b]$ and define

$$l_{\mathcal{P}} := \sum_{i=1}^{n} m_i \mathbb{I}_{(t_{i-1},t_i]} \quad \text{and} \quad u_{\mathcal{P}} := \sum_{i=1}^{n} M_i \mathbb{I}_{(t_{i-1},t_i]},$$

where

$$m_i := \inf\{\, f(x) : x \in [t_{i-1}, t_i] \,\} \quad \text{and} \quad M_i := \sup\{\, f(x) : x \in [t_{i-1}, t_i] \,\}.$$

Notice that

$$\int l_{\mathcal{P}} = L(f, \mathcal{P}) \quad \text{and} \quad \int u_{\mathcal{P}} = U(f, \mathcal{P}).$$

We can choose sequences of partitions $\{\, \mathcal{Q}_k \,\}$ and $\{\, \mathcal{R}_k \,\}$ such that

$$\lim_k L(f, \mathcal{Q}_k) = \mathcal{R}\underline{\int_a^b} f \quad \text{and} \quad \lim_k U(f, \mathcal{R}_k) = \mathcal{R}\overline{\int_a^b} f. \tag{2}$$

Let $\mathcal{P}_k = \{\, a = t_0 < \cdots < t_n = b \,\}$ be a partition which is refinement of the partitions $\mathcal{Q}_k$ and $\mathcal{R}_k$ as well as $\mathcal{P}_{k-1}$, and which also has the property that $\|\mathcal{P}_k\| := \max(t_i - t_{i-1}) < \frac{1}{k}$. Since $\mathcal{P}_k$ is a refinement of both $\mathcal{Q}_k$ and $\mathcal{R}_k$, (2) holds with $\mathcal{Q}_k$ and $\mathcal{R}_k$ each replaced by $\mathcal{P}_k$. Since $\mathcal{P}_{k+1}$ is a refinement of $\mathcal{P}_k$, it follows that

$$l_{\mathcal{P}_{k+1}} \geq l_{\mathcal{P}_k} \quad \text{and} \quad u_{\mathcal{P}_{k+1}} \leq u_{\mathcal{P}_k}.$$

Therefore we obtain bounded measurable functions l and u on $[a,b]$ by

$$l := \sup_k l_{\mathcal{P}_k} = \lim_k l_{\mathcal{P}_k} \quad \text{and} \quad u := \inf_k u_{\mathcal{P}_k} = \lim_k u_{\mathcal{P}_k}.$$

Clearly

$$l \le f \le u.$$

Since bounded functions are Lebesque integrable on $[a,b]$ and since $u = \lim_k u_{\mathcal{P}_k}$ and $l = \lim_k l_{\mathcal{P}_k}$, the Lebesgue Dominated Convergence Theorem implies that

$$\int l = \mathcal{R}\underline{\int_a^b} f \quad \text{and} \quad \int u = \mathcal{R}\overline{\int_a^b} f.$$

Now if f is Riemann integrable, the upper and lower integrals coincide and we have

$$\int (u - l) = 0.$$

Since $u - l \ge 0$, this implies that $l = f = u$ a.e. Since Lebesque measure is complete, f is measurable, and

$$\mathcal{R}\int_a^b f = \int f.$$

This proves the first part.

To prove the second assertion, first observe that if $x \in [a,b]$ and if $0 < \delta < \delta'$, then

$$\sup\{\, f(y) : |y - x| \le \delta \,\} \le \sup\{\, f(y) : |y - x| \le \delta' \,\}.$$

It follows that

$$\lim_{\delta \to 0} \sup\{\, f(y) : |y - x| \le \delta \,\} = \inf_{\delta > 0} \sup\{\, f(y) : |y - x| \le \delta \,\}. \tag{3}$$

Thus we get a well defined function H on $[a,b]$ by setting $H(x)$ equal to (3). Similarly, we can define h on $[a,b]$ by

$$h(x) := \lim_{\delta \to 0} \inf\{\, f(y) : |y - x| \le \delta \,\} = \sup_{\delta > 0} \inf\{\, f(y) : |y - x| \le \delta \,\}. \tag{4}$$

We clearly have $h(x) \le f(x) \le H(x)$ for all $x \in [a,b]$.

Suppose that f is continuous at x. Then given $\epsilon > 0$ there is a $\delta > 0$ such that whenever $|y - x| \le \delta$ we have $|f(y) - f(x)| < \epsilon$. This is the same as

$$f(x) - \epsilon < f(y) < f(x) + \epsilon. \tag{5}$$

It follows from (3) and (5) that $H(x) < f(x) + \epsilon$. Since ϵ is arbitrary, we must have $H(x) \le f(x)$. Thus $H(x) = f(x)$ in the event that f is continuous at x. Similarly, combining (3) and (4) shows that $h(x) > f(x) - \epsilon$ for any $\epsilon > 0$. Thus forces $h(x) = f(x)$ when f is continuous at x. In particular, $H(x) = h(x)$ if f is continuous at x.

Now suppose that $H(x) = h(x)$. Note that the common value must be $f(x)$. Thus given $\epsilon > 0$, there is — in view of (3) and (4) — a $\delta > 0$ such that

$$f(x) + \epsilon = H(x) + \epsilon > \sup\{\, f(y) : |y - x| \le \delta \,\} \quad \text{and} \tag{6}$$

$$f(x) - \epsilon = h(x) - \epsilon < \inf\{\, f(y) : |y - x| \le \delta \,\}. \tag{7}$$

Thus if $|y - x| < \delta$, then we have

$$f(x) - \epsilon < f(y) < f(x) + \epsilon \quad \text{or} \quad |f(y) - f(x)| < \epsilon.$$

This shows that f is continuous at x if and only if $H(x) = h(x)$.[1]

If $\mathcal{P} = \{\, a = t_0 < \cdots < t_n = b \,\}$ is any partition of $[a, b]$ and if $x \notin \mathcal{P}$, then there is a $\delta > 0$ such that $\{\, y : |y - x| \le \delta \,\} \cap \mathcal{P} = \emptyset$. In particular, $\{\, y : |y - x| \le \delta \,\} \subset (t_{i-1}, t_i)$ for some i, and

$$M_i \ge \sup\{\, f(y) : |y - x| \le \delta \,\}.$$

It follows that $u_{\mathcal{P}}(x) \ge H(x)$ for all $x \notin \mathcal{P}$. Now let

$$N := \bigcup_k \mathcal{P}_k.$$

Then N is countable, and therefore has Lebesgue measure 0. Furthermore if $x \notin N$, then

$$u(x) := \inf u_{\mathcal{P}_k}(x) \ge H(x).$$

On the other hand, given $x \notin N$ and $\epsilon > 0$, there is a $\delta > 0$ such that

$$H(x) + \epsilon > \sup\{\, f(y) : |y - x| \le \delta \,\}.$$

[1] This is the first of Folland's suggested "Lemmas".

Pick k such that $\frac{1}{k} < \delta$. Since $x \notin \mathcal{P}_k$, $x \in (t_{i-1}, t_i)$ for some subinterval in $\mathcal{P}_k$. Since $\|\mathcal{P}_k\| < \frac{1}{k}$, $M_i \leq \sup\{\, f(y) : |y - x| \leq \delta \,\}$ and

$$H(x) + \epsilon > u_{\mathcal{P}_k}(x) \geq u(x).$$

Since ϵ was arbitrary, we conclude that $H(x) = u(x)$ for all $x \notin N$. In particular, H is measurable and

$$\int H = \mathcal{R}\overline{\int_a^b} f.$$

A similar argument implies that $h(x) = l(x)$ for all $x \notin N$. Thus h is measurable and[2]

$$\int h = \mathcal{R}\underline{\int_a^b} f.$$

Now if f is continuous almost everywhere, it follows that $H = h$ a.e. Thus the upper and lower Riemann integrals must be equal and f is Riemann integrable. On the other hand, if f is Riemann integrable, the upper and lower integrals are equal and

$$\int (H - h) = 0.$$

Since $H - h \geq 0$, we must have $H = h$ a.e. It follows that f is continuous almost everywhere. $\square$

[2] This is essentially Folland's Lemma (b).