

Math 73/103 - Fall 2016

Problem set 2

1. Page 32 of the text, problem #6. (Note that we have already shown that \mathfrak{M} is a σ -algebra so there is no need to show it again.)
2. Page 32 of the text, problem #7.
3. Page 32 of the text, problem #10.
4. Page 32 of the text, problem #12. (This is easy if f is bounded.)
5. Suppose that Y is a topological space and that \mathfrak{M} is a σ -algebra in Y containing all the Borel sets. Suppose in addition, μ is a measure on (Y, \mathfrak{M}) such that for all $E \in \mathfrak{M}$ we have

$$\mu(E) = \inf\{\mu(V) : V \text{ is open and } E \subset V\}. \quad (1)$$

Suppose also that

$$Y = \bigcup_{n=1}^{\infty} Y_n \quad \text{with } \mu(Y_n) < \infty \text{ for all } n \geq 1. \quad (2)$$

One says that μ is a σ -finite outer regular measure on (Y, \mathfrak{M}) .

- (a) Show that Lebesgue measure m is a σ -finite outer regular measure on $(\mathbf{R}, \mathfrak{M})$.
 - (b) Suppose E is a μ -measurable subset of Y .
 - (i) Given $\varepsilon > 0$, show that there is an open set $V \subset Y$ and a closed set $F \subset Y$ such that $F \subset E \subset V$ and $\mu(V \setminus F) < \varepsilon$.
 - (ii) Show that there is a G_δ -set $G \subset Y$ and a F_σ -set $A \subset Y$ such that $A \subset E \subset G$ and $\mu(G \setminus A) = 0$.
 - (c) Argue that $(\mathbf{R}, \mathfrak{M}, m)$ is the completion of the restriction of Lebesgue measure to the Borel sets in \mathbf{R} .
6. Let m be Lebesgue measure on \mathbf{R} and suppose that E is a set of finite measure. Given $\varepsilon > 0$, show that there is a finite disjoint union F of open intervals such that $m(E \Delta F) < \varepsilon$ where $E \Delta F := (E \setminus F) \cup (F \setminus E)$ is the symmetric difference. (This illustrates the first of Littlewood's three principles: "Every Lebesgue measurable set is nearly a disjoint union of open intervals".)

7. Let (X, \mathfrak{M}, μ) be a measure space, and let $(X, \mathfrak{M}_0, \mu_0)$ be its completion.

- (a) Let $f : X \rightarrow \mathbf{C}$ be a μ_0 -measurable function and assume that $g : X \rightarrow \mathbf{C}$ is a μ -measurable function such that $f = g$ a.e. $[\mu_0]$. Is there necessarily a μ -null set N such that $f(x) = g(x)$ for all $x \notin N$?
- (b) If $f : X \rightarrow \mathbf{C}$ is μ_0 -measurable, show that there is a μ -measurable function $g : X \rightarrow \mathbf{C}$ such that $f = g$ a.e. $[\mu_0]$.
- (c) What does this result say about Lebesgue measurable functions and Borel functions on \mathbf{R} ? (Compare with problem #14 on page 59 of the text.)

8. Suppose that (X, \mathfrak{M}, μ) is a measure space. Recall that $E \in \mathfrak{M}$ is called σ -finite if E is the countable union of sets of finite measure. Let $f \in \mathcal{L}^1(\mu)$.

- (a) Show that $\{x \in X : f(x) \neq 0\}$ is σ -finite.
- (b) Suppose that $f \geq 0$. Show that there are (measurable) simple functions φ_n such that $\varphi_n \nearrow f$ everywhere and there is a single σ -finite set outside of which the φ_n vanish.
- (c) Given $\varepsilon > 0$ show that there is simple function such that

$$\int_X |f - \varphi| d\mu < \varepsilon.$$

- (d) If $(X, \mathfrak{M}, \mu) = (\mathbf{R}, \mathfrak{M}, m)$ is Lebesgue measure, show that we can take the simple function φ in part (c) to be a step function — that is, a finite linear combination of characteristic functions of *intervals*.