# Math 73/103 - Fall 2016
# Problem set 3

1. Suppose that $Y$ is a topological space and that $\mathfrak{M}$ is a $\sigma$-algebra in $Y$ containing all the Borel sets. Suppose in addition, $\mu$ is a measure on $(Y, \mathfrak{M})$ such that for all $E \in \mathfrak{M}$ we have

$$\mu(E) = \inf\{\, \mu(V) : V \text{ is open and } E \subset V \,\}. \tag{1}$$

Suppose also that

$$Y = \bigcup_{n=1}^{\infty} Y_n \quad \text{with } \mu(Y_n) < \infty \text{ for all } n \geq 1. \tag{2}$$

One says that $\mu$ is a *$\sigma$-finite outer regular measure* on $(Y, \mathfrak{M})$.

(a) Show that Lebesgue measure $\lambda$ is a $\sigma$-finite outer regular measure on $(\mathbf{R}, \Lambda^*)$.

(b) Suppose $E$ is a $\mu$-measurable subset of $Y$.

(i) Given $\varepsilon > 0$, show that there is an open set $V \subset Y$ and a closed set $F \subset Y$ such that $F \subset E \subset V$ and $\mu(V \setminus F) < \varepsilon$.

(ii) Show that there is a $G_\delta$-set $G \subset Y$ and a $F_\sigma$-set $A \subset Y$ such that $A \subset E \subset G$ and $\mu(G \setminus A) = 0$.

(c) Argue that $(\mathbf{R}, \Lambda^*, \lambda)$ is the completion of the restriction of Lebesgue measure to the Borel sets in $\mathbf{R}$.

2. Let $\lambda$ be Lebesgue measure on $\mathbf{R}$ and suppose that $E$ is a set of finite measure. Given $\varepsilon > 0$, show that there is a finite *disjoint* union $F$ of open intervals such that $\lambda(E \triangle F) < \varepsilon$ where $E \triangle F := (E \setminus F) \cup (F \setminus E)$ is the symmetric difference. (This illustrates the first of Littlewood's three principles: "Every Lebesgue measurable set is nearly a disjoint union of open intervals".)

3. Suppose that $(X, \mathfrak{M}, \mu)$ is a measure space. Recall that $E \in \mathfrak{M}$ is called $\sigma$-finite if $E$ is the countable union of sets of finite measure. Let $f \in \mathcal{L}^1(\mu)$.

(a) Show that $\{\, x \in X : f(x) \neq 0 \,\}$ is $\sigma$-finite.

(b) Suppose that $f \geq 0$. Show that there are (measurable) simple functions $\varphi_n$ such that $\varphi_n \nearrow f$ everywhere and there is a single $\sigma$-finite set outside of which the $\varphi_n$ vanish.

(c) Given $\varepsilon > 0$ show that there is a simple function such that $\displaystyle\int_X |f - \varphi| \, d\mu < \varepsilon$.

(d) If $(X, \mathfrak{M}, \mu) = (\mathbf{R}, \Lambda^*, \lambda)$ is Lebesgue measure, show that we can take the simple function $\varphi$ in part (c) to be a step function — that is, a finite linear combination of characteristic functions of *intervals*.

4. Suppose that $f \in \mathcal{L}^1(\mathbf{R}, \Lambda^*, \lambda)$ is a Lebesgue integrable function on the real line. Let $\varepsilon > 0$. Show that there is a continuous function $g$ that vanishes outside a bounded interval such that $\|f - g\|_1 < \varepsilon$. (Hint: this is easy if $f$ is the characteristic function of a bounded interval: draw a picture.)

Another of Littlewood's Principles is that a pointwise convergent sequence of functions is nearly uniformly convergent. This is also known as "Egoroff's Theorem".

5. Prove Egoroff's Theorem: Suppose that $(X, \mathfrak{M}, \mu)$ is a finite measure space — that is, $\mu(X) < \infty$. Suppose that $\{f_n\}$ is a sequence of measurable functions converging almost everywhere to a measurable function $f : X \to \mathbf{C}$. Then for all $\varepsilon > 0$ there is a set $E \in \mathfrak{M}$ such that $\mu(X \setminus E) < \varepsilon$ and $f_n \to f$ uniformly on $E$.

Some suggestions:

(a) There is no harm in assuming that $f_n \to f$ everywhere.

(b) Let $E_n(k) = \bigcup_{m=n}^{\infty} \{ x \in X : |f_m(x) - f(x)| \geq \frac{1}{k} \}$.

(c) Show that $\lim_{n\to\infty} \mu(E_n(k)) = 0$. (You need $\mu(X) < \infty$ here.)

(d) Fix $\varepsilon > 0$ and $k$. Choose $n_k \geq n$ so that $\mu(E_{n_k}(k)) < \frac{\varepsilon}{2^{-k}}$, and let $E = \displaystyle\bigcup_{k=1}^{\infty} E_{n_k}(k)$.

6. Suppose that $\rho$ is a premeasure on an algebra $\mathcal{A}$ of sets in $X$. Let $\rho^*$ be the associated outer measure.

(a) Show that $\rho^*(E) = \rho(E)$ for all $E \in \mathcal{A}$.

(b) If $\mathfrak{M}^*$ is the $\sigma$-algebra of $\rho^*$-measurable sets, show that $\mathcal{A} \subset \mathfrak{M}^*$.