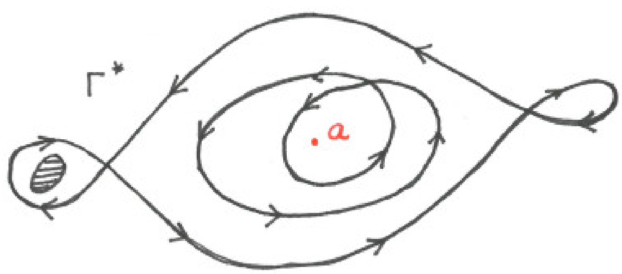
NAME: GRADE:

Math 73/103

Final Examination: Complex Analysis

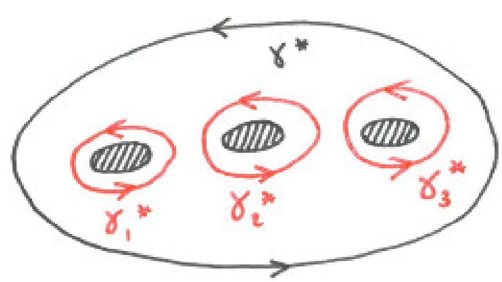
1. For this problem only, short non-technical explanations will be accepted.

a. Let Γ be the cycle represented below (it has two connected components).



$\mathrm{Ind}_\Gamma(a) =$

b. Are the cycles γ and $\gamma_1 + \gamma_2 + \gamma_3$ homologous in the complement Ω of the shaded area? Why?



2. a. Prove that a bounded entire function $f \in H(\mathbb{C})$ is necessarily constant.
(Do **not** quote Liouville's Theorem.)

b. Use (a) to prove that every non-constant polynomial has a complex root.

3. Let Ω be a simply connected region, $a \in \Omega$ and f a holomorphic function on Ω. For $z \in \Omega$, define

$$F(z) = \int_{\gamma(z)} f(w)\, dw$$

where $\gamma(z)$ is a path from a to z.

a. Recall why F is well-defined.

b. Prove that $F' = f$.

4. Shwarz's Lemma.

a. Let u be a holomorphic function on a domain Ω containing a closed disk $\overline{D}$. Prove the existence of z_0 on the boundary of $\overline{D}$ such that

$$|u(z)| \leq |u(z_0)|$$

for all z in the interior D of the disk.

b. Let f be a holomorphic function from $D_1(0) = \{z \in \mathbb{C} \ : \ |z| < 1\}$ to itself satisfying $f(0) = 0$. Prove that

$$|f(z)| \leq |z|$$

for all $z \in D_0(1)$ and that $|f'(0)| \leq 1$.

Hint: study the function $z \mapsto \dfrac{f(z)}{z}$ *on* $D_r(0)$ *for* $r < 1$.

5. Gutzmer's Formula.

Let $f(z) = \sum_{n \geq 0} a_n z^n$ be an entire function and $\gamma(t) = re^{it}$ for $0 \leq t \leq 2\pi$ with $r > 0$.

a. Justify that

$$\int_0^{2\pi} |f(re^{it})|^2 \, dt = \sum_{n \geq 0} \int_0^{2\pi} \frac{f(re^{it})\overline{a_n} r^n}{e^{int}} \, dt.$$

b. Prove Gutzmer's Formula:

$$\int_0^{2\pi} |f(re^{it})|^2 \, dt = 2\pi \sum_{n \geq 0} |a_n|^2 r^{2n}.$$

Hint: Cauchy's Formula for a_n.

6. Let f be a holomorphic function with real and imaginary parts u and v seen as functions of the polar coordinates, so that

$$f(re^{i\theta}) = u(r,\theta) + iv(r,\theta).$$

The goal of this problem is to prove that on $\mathbb{C} \setminus \{0\}$,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{and} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

a. Briefly describe your strategy.

b. Do it.

7. Evaluate, for $a > 0$, the integral

$$\mathcal{I} = \int_{-\infty}^{+\infty} \frac{\cos x}{x^2 + a^2}\, dx.$$

You are not required to prove that $\mathcal{I}$ absolutely convergent.

Suggestion: consider the function of the complex variable $f(z) = \dfrac{e^{iz}}{z^2 + a^2}$ *and a contour* Γ_R *consisting of the line segment* $[-R, R]$ *and a half-circle.* ***Draw a picture***.

[EXTRA SPACE - p.1]

[EXTRA SPACE - p.2]