Math 73/103: Measure Theory and Complex Analysis
Fall 2018 - Homework 4

The following problems reprise some of the fundamental results about functions of a complex variable covered in elementary courses but not covered in Chapter 10 of [Rudin]. Most of this material — with perhaps the exception of Question 4 — are part of the early chapters in basic texts such as Conway, Brown & Churchill or Saff & Snider.

Recall: Let Ω be a domain in $\mathbb{C}$ and assume that $f : \Omega \to \mathbb{C}$ is a function. Of course, we can view Ω as an open subset of $\mathbb{R}^2$ and define $u, v : \Omega \to \mathbb{R}$ by

$$u(x,y) := \operatorname{Re}\bigl(f(x+iy)\bigr) \quad \text{and} \quad v(x,y) = \operatorname{Im}\bigl(f(x+iy)\bigr)$$

We say that the *Cauchy-Riemann Equations hold at* $z_0 = x_0 + iy_0$ if the partial derivatives of u and v exist at (x_0, y_0) and

$$u_x(x_0,y_0) = v_y(x_0,y_0) \quad \text{and} \quad u_y(x_0,y_0) = -v_x(x_0,y_0). \tag{CRE}$$

We often abuse notation slightly, and say that (CRE) amounts to $f_y(z_0) = if_x(z_0)$. (Just to be specific, $f_x(x_0 + iy_0) := u_x(x_0,y_0) + iv_x(x_0,y_0)$.)

1. Suppose that Ω is a *region* in $\mathbb{C}$, and that $f \in \mathcal{H}(\Omega)$. Show that if $f'(z) = 0$ for all $z \in \Omega$, then f is constant.
Hint: You can prove for yourself or use without proof that if $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is such that $u_x(x,y) = u_y(x,y)$ for all $(x,y) \in \Omega$ then u is constant — provided Ω is a region.

2. Suppose that Ω is a region and $f \in \mathcal{H}(\Omega)$. Show that if f is real-valued in Ω, then f is constant.

3. Suppose that Ω is a region and $f \in \mathcal{H}(\Omega)$. Suppose that $z \mapsto |f(z)|$ is constant on Ω. Show that f must be constant. **Hint:** Consider $|f(z)|^2$.

We let f, u, v and Ω be as above. Define

$$F : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{by} \quad F(x,y) = \bigl(u(x,y), v(x,y)\bigr).$$

Recall that F is differentiable at $(x_0,y_0) \in \Omega$ if there is a linear function $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\lim_{(h,k)\to(0,0)} \frac{\|F(x_0+h, y_0+k) - F(x_0,y_0) - L(h,k)\|}{\|(h,k)\|} = 0,$$

in which case, the partials of u and v must exist and L is given by the Jacobian Matrix

$$[L] = \begin{pmatrix} u_x(x_0,y_0) & u_y(x_0,y_0) \\ v_x(x_0,y_0) & v_y(x_0,y_0) \end{pmatrix}.$$

(Here $\|(x,y)\| = \sqrt{x^2+y^2} = |x+iy|$.)

4. Let f, F, u, v and Ω be as above. Let $z_0 = x_0 + iy_0 \in \Omega$. Show that $f'(z_0)$ exists if and only if the Cauchy-Riemann equations hold at z_0 and F is differentiable at (x_0, y_0).
Hint: if we let $z = h + ik$ and if T is given by the matrix

$$[T] = \begin{pmatrix} u_x(x_0, y_0) & -v_x(x_0, y_0) \\ v_x(x_0, y_0) & u_x(x_0, y_0) \end{pmatrix},$$

then

$$\|F(x_0 + h, y_0 + k) - F(x_0, y_0) - T(h, k)\| = |f(z + z_0) - f(z_0) - \omega z|,$$

where $\omega = u_x(x_0, y_0) + iv_x(x_0, y_0) = f_x(z_0)$.)

Question #4 has an important corollary. We learn in multivariable calculus, that F is differentiable at (x_0, y_0) if the partial derivatives of u and v exist in a neighborhood of (x_0, y_0) and are continuous at (x_0, y_0). Hence we get as a corollary of Question #4, with f, u and v defined as above, that if u and v have continuous partial derivatives in a neighborhood of (x_0, y_0) and if the Cauchy-Riemann equations hold at z_0, then $f'(z_0)$ exists. Use this observation in Question #5.

5. Define $\exp : \mathbb{C} \to \mathbb{C}$ by $\exp(x + iy) = e^x\bigl(\cos(y) + i\sin(y)\bigr)$. Show that $\exp \in H(\mathbb{C})$ and $\exp'(z) = \exp(z)$ for all $z \in \mathbb{C}$.

If Ω is open in $\mathbb{C}$ or $\mathbb{R}^2$, then we say $u : \Omega \to \mathbb{R}$ is *harmonic* if it has continuous second partial derivatives and if it is a solution to Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{L}$$

6. Suppose that $f \in \mathcal{H}(\Omega)$. Let $u(x, y) = \mathrm{Re}\bigl(f(x + iy)\bigr)$. Assuming u has continuous second partials, show that u is harmonic in Ω.

7. Suppose that $u : \Omega \to \mathbb{R}$ is harmonic. We say $v : \Omega \to \mathbb{R}$ is a *harmonic conjugate* for u if $f(x + iy) = u(x, y) + iv(x, y)$ defines a holomorphic function on Ω. Find all harmonic conjugates for $u(x, y) = 2xy$.

For the purposes of this assignment only, we'll call a region Ω a *SC region* if every $f \in \mathcal{H}(\Omega)$ has an antiderivative in Ω. For example, we have shown in lecture that every convex region is a SC region. Later, I hope that we'll see that any simply connected region is a SC region. In fact, a region is a SC-region if and only if it is simply connected.

8. Suppose that Ω is a SC region and that u is harmonic in Ω. Show that u has a harmonic conjugate in Ω.
Hint: we need to find a function $f \in \mathcal{H}(\Omega)$ such that $u = \mathrm{Re}(f)$. However, consider $g = u_x - iu_y$. Show that $g \in \mathcal{H}(\Omega)$ and consider an anti-derivative f for g in Ω. As in Question 1, you may use without proof the fact that if $w : \Omega \to \mathbb{R}$ is continuous and $w_x \equiv 0 \equiv w_y$ in Ω, then w is constant.

If $u = \mathrm{Re}(f)$, then $u_x = \mathrm{Re}(f')$ and $u_y = \mathrm{Re}(-if')$. Thus, it is a consequence of Question #8 (and the deep result that $f \in \mathcal{H}(\Omega)$ implies f is analytic) that every harmonic function has continuous partial derivatives of all orders.

9. Just as in Question #5, we shall write $\exp(z)$ in place of e^z. Suppose that Ω is a SC region and that $0 \notin \Omega$. Then show there is a $f \in \mathcal{H}(\Omega)$ such that

$$\exp\bigl(f(z)\bigr) = z.$$

We call f a *branch of* $\log(z)$ *in* Ω. (Hint: start by letting f be an antiderivative of $1/z$. and recall that $\exp(z) = a$ has infinitely many solutions for all $a \neq 0$.)

10. Show that $f(z) = 1/z$ can't have an antiderivative in the punctured complex plane $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Conclude that there is no (holomorphic) branch of $\log z$ in $\mathbb{C}^*$.