Math 73/103: Measure Theory and Complex Analysis
Fall 2019 - Homework 2

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1. Page 32 of Rudin, problem #6. (Note that we have already shown that $\mathcal{M}$ is a σ-algebra so there is no need to show it again.)

ANS: We already know $\mathcal{M}$ is a σ-algebra. Let $\{E_i\}$ be a countable pairwise disjoint family of measurable subsets with $E := \bigcup_i E_i$. If all the E_i are countable then so is E. Thus we clearly have

$$\mu(E) = \sum_i \mu(E_i).$$

On the other hand, if one the E_i — say E_k is uncountable, then E_k^c is countable and contains all the other E_i with $i \neq k$. Thus E is uncountable, $\mu(E) = 1$ and

$$\sum_i \mu(E_i) = \mu(E_k) = 1.$$

Thus μ is a measure.

The key to the rest of the problem is to realize that $f : (X, \mathcal{M}) \to \mathbb{C}$ is measurable if and only if f is constant μ-almost everywhere; that is, f is measurable if and only if there is a $c \in \mathbb{C}$ such that $f^{-1}(X \setminus \{c\})$ is countable. Of course, if this assertion is correct, then f is equal to the constant function $g(x) = c$ almost everywhere and

$$\int_X f \, d\mu = \int_X c \, d\mu = c\mu(X) = c.$$

It is fairly clear that if f is constant almost everywhere, then f is measurable. So, assume that f is measurable. Then for any open set V, either $f^{-1}(V)$ is uncountable or $f^{-1}(V)^c = f^{-1}(V^c)$ is uncountable. Let $\{V_n\}$ be a *countable* basis for the topology of $\mathbb{C}$. In view of the above, let

$$B_n := \begin{cases} V_n & \text{if } f^{-1}(V_n) \text{ is uncountable, and} \\ V_n^c & \text{if } f^{-1}(V_n^c) \text{ is uncountable.} \end{cases}$$

Let $A = \bigcap B_n$. I claim that A can consist of at most one point. If $x \neq y$, then there is a n such that $x \in V_n$ and $y \in V_n^c$. Thus at most one of x and y belong to B_n. Thus at most one of x and y can belong to A. Now it will suffice to see that $f^{-1}(A)$ is uncountable. (This implies its complement is countable.) For this, it suffices to see that $\mu(f^{-1}(A)) = 1$.

But since $C \cup D$ is the disjoint union of $C \setminus D$, $C \cap D$ and $D \setminus C$, it follows that if both $f^{-1}(C)$ and $f^{-1}(D)$ are uncountable, then so is $f^{-1}(C \cap D)$. But

$$A = \bigcap_n B_n = \bigcap_n F_n \quad \text{where, } F_n = B_1 \cap \cdots \cap B_n.$$

Then $f^{-1}(F_n)$ is uncountable and

$$\mu(f^{-1}(A)) = \lim_n \mu(f^{-1}(F_n)) = 1.$$

This completes the proof.

2. Page 32 of Rudin, problem #7.

ANS: Note that $f_n(x) = |f_n(x)| \leq f_1(x)$ for all $x \in X$, and $f_1 \in \mathcal{L}^1(\mu)$. Therefore, the conclusion follows from the LDCT. For a counterexample, take $f_k := \mathbb{I}_{[k,\infty)}$. Then $f_k \searrow 0$, but all the f_k have infinite integrals.

3. Page 32 of Rudin, problem #10.

ANS: Since constant functions are summable if $\mu(X) < \infty$, use the LDCT on the sequence $g_n = |f_n - f|$ together with the observation that f_n and $f_n - f$ integrable implies f is too.[1]

For a counterexample when $\mu(X) = \infty$, consider Lebesgue measure on $\mathbb{R}$ and set $f_n = \frac{1}{n}\mathbb{I}_{[-n,n]}$. Then $f_n \to 0$ uniformly, while $\int_{\mathbb{R}} f_n \, du = 2$ for all n.

4. Page 32 of Rudin, problem #12. (This is easy if f is bounded.)

ANS: First notice that the conclusion is obvious if f is bounded[2]. In general, let $f_n = \min\{|f|, n\}$. Since $f_n \nearrow |f|$, the MCT implies that $\int_X f_n \, d\mu \nearrow \int_X |f| \, d\mu$. In particular, we can choose N such that

$$\left| \int_X f_N \, d\mu - \int_X |f| \, d\mu \right| < \frac{\varepsilon}{2}.$$

Now since f_N is bounded, choose $\delta > 0$ so that $\mu(E) < \delta$ implies that $\int_E f_N \, d\mu < \varepsilon/2$. The point being that

$$\int_E |f| \, d\mu \leq \left| \int_E f_N \, d\mu \right| + \left| \int_X (|f| - f_N) \, d\mu \right| < \varepsilon.$$

(We've used $|f| \geq f_N$ for the second to last inequality.)

5. Suppose that Y is a topological space and that $\mathcal{M}$ is a σ-algebra in Y containing all the Borel sets. Suppose in addition, μ is a measure on $(Y, \mathcal{M})$ such that for all $E \in \mathcal{M}$ we have

$$\mu(E) = \inf\{\, \mu(V) : V \text{ is open and } E \subset V \,\}. \tag{1}$$

Suppose also that

$$Y = \bigcup_{n=1}^{\infty} Y_n \quad \text{with } \mu(Y_n) < \infty \text{ for all } n \geq 1. \tag{2}$$

One says that μ is a *σ-finite outer regular measure* on $(Y, \mathcal{M})$.

(a) Show that Lebesgue measure m is a σ-finite outer regular measure on $(\mathbb{R}, \mathcal{M})$.

ANS: Since (1) is obviously satisfied if $m(E) = \infty$, we can assume that $m(E) < \infty$. If $\varepsilon > 0$, then by definition of m (as the restriction of m^*), there are open intervals $\{I_n\}$) such that

$$E \subset \bigcup_n I_n \quad \text{and} \quad m(E) + \varepsilon > \sum_n \ell(I_n).$$

[1] Alternatively, you can show that the $\{\, f_n \,\}$ are *uniformly bounded*; that is, there exists M such that $\|f_n\|_\infty \leq M$ for all n. However, you must prove this. By assumption, we only know that for each n, $M_n := \|f_n\|_\infty < \infty$. But by assumption, there is a N such that $n \geq N$ implies $\|f_n - f\|_\infty < 1$. It follows that $\|f\|_\infty \leq \|f_N\|_\infty + 1$ and *for all* n

$$\|f_n\|_\infty \leq M := \max\{\, \|f\|_1, \dots, \|f_N\|_\infty, \|f_N\|_\infty + 2 \,\}.$$

Now we can apply the LDCT with $g \equiv M$.

[2] This technique is used quite often—reduce the problem to a simpler situation (e.g., a characteristic function, simple function, or, as here, a bounded function.

But $V := \bigcup_n I_n$ is an open set containing E and

$$m(E) + \varepsilon > \sum_n \ell(I_n) = \sum_n m(I_n) \geq m(V) \geq m(E).$$

This implies (1). Since $\mathbb{R} = \bigcup_n [-n, n]$, Lebesgue measure is also σ-finite. This proves (a).

(b) Suppose E is a μ-measurable subset of Y.

(i) Given $\varepsilon > 0$, show that there is an open set $V \subset Y$ and a closed set $F \subset Y$ such that $F \subset E \subset V$ and $\mu(V \setminus F) < \varepsilon$.

ANS: Suppose $\mu(E) < \infty$. Then in view of (1), there is an open set $V \supset E$ such that $\mu(V) - \mu(E) < \varepsilon/2$. Since $\mu(E) < \infty$, $\mu(V \setminus E) < \varepsilon/2$. Now in general, $X = \bigcup_n X_n$ with $\mu(X_n) < \infty$ for each n. Let $E_n = E \cap X_n$. Then there are open sets $V_n \subset E_n$ such that $\mu(V_n \setminus E_n) < \frac{\varepsilon}{2^{n=1}}$. Let $V = \bigcup V_n$. Then V is open and contains E. Furthermore,

$$\mu(V \setminus E) = \mu\Bigl(\bigcup V_n \setminus \bigcup E_n\Bigr) \leq \mu\Bigl(\bigcup_n (V_n \setminus E_n)\Bigr) \leq \varepsilon/2.$$

But the above reasoning shows that there is an open set W containing E^c such that $\mu(W \setminus E^c) < \varepsilon/2$. Then $F = W^c$ is a closed subset of E, and $\mu(E \setminus F) < \varepsilon/2$. Then, since $V \setminus F = V \setminus E \cup E \setminus F$, we have $\mu(V \setminus F) < \varepsilon$ as required.

(c) Argue that $(\mathbb{R}, \mathcal{M}, m)$ is the completion of the restriction of Lebesgue measure to the Borel sets in $\mathbb{R}$.

ANS: Let $\mathcal{B} = \mathcal{B}(\mathbb{R})$ be the Borel sets in $\mathbb{R}$. Let $(\mathbb{R}, \mathcal{B}_0, m_0)$ be the completion. Since G_δ sets and F_σ sets are Borel, part (b)(ii) shows that $\mathcal{M} \subset \mathcal{B}_0$. But if $E \in \mathcal{B}_0$, then $E = B \cup N$ where B is Borel and N is a subset of a Borel m-null set. Since Lebesgue measure is complete, $N \in \mathcal{M}$, and hence, $E \in \mathcal{M}$. Thus $\mathcal{M} = \mathcal{B}_0$ and it is clear that $m = m_0$.

6. Let m be Lebesgue measure on $\mathbb{R}$ and suppose that E is a set of finite measure. Given $\varepsilon > 0$, show that there is a finite *disjoint* union F of open intervals such that $m(E \triangle F) < \varepsilon$ where $E \triangle F := (E \setminus F) \cup (F \setminus E)$ is the symmetric difference. (This illustrates the first of Littlewood's three principles: "Every Lebesgue measurable set is nearly a disjoint union of open intervals".)

ANS: In view of problem 5a, there is an open set $V \subset \mathbb{R}$ containing E such that $m(V \setminus E) < \varepsilon/2$. But V is a countable *disjoint union* of intervals: $V = \bigcup_n I_n$. Since $\mu(E) < \infty$, we must also have $\mu(V) < \infty$ and then

$$\infty > m(V) = \sum_{n=1}^{\infty} m(I_n),$$

there is a N such that $\sum_{n>N} m(I_n) < \varepsilon/2$. Let $F = \bigcup_{n=1}^{N} I_n$. (Then F is a disjoint union of intervals.) Also

$$m(E \Delta F) = m(E \setminus F) + m(F \setminus E) \leq m(V \setminus F) + m(V \setminus E) < \varepsilon.$$

7. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $(X, \mathcal{M}_0, \mu_0)$ be its completion.

(a) Let $f : X \to \mathbb{C}$ be a μ_0-measurable function and assume that $g : X \to \mathbb{C}$ is a μ-measurable function such that $f = g$ a.e. $[\mu_0]$. Is there necessarily a μ-null set N such that $f(x) = g(x)$ for all $x \notin N$?

ANS: If g is μ-measurable, then $\{ x \in X : f(x) \neq g(x) \}$ is only guaranteed to belong to $\mathcal{M}_0$. But if it is a μ_0-null set then it is contained in a μ-null set $N \in \mathcal{M}$.

(b) If $f : X \to \mathbb{C}$ is μ_0-measurable, show that there is a μ-measurable function $g : X \to \mathbb{C}$ such that $f = g$ a.e. $[\mu_0]$.

ANS: Clearly, it suffices to consider only functions $f : X \to [0, \infty)$. I claim it is enough to prove that the result is true for simple functions. In view of (a), this means that given any μ_0-measurable simple function s, there is a μ-measurable simple function s' which agrees with s off a μ-null set in $\mathcal{M}$. Recall that there are nonnegative μ_0-measurable simple functions $s_n \nearrow f$. If there are nonnegative μ-measurable simple functions s'_n and null sets $N_n \in \mathcal{M}$ so that $s'_n = s_n$ off N_n, then $s'_n \nearrow f$ except possibly on the null set $N = \bigcup N_n \in \mathcal{M}$. Replacing the s'_n by $s''_n = \mathbb{I}_{X \setminus N} \cdot s'_n$, then the sequence $\{ s''_n \}$ converges everywhere to a function g which is necessarily μ-measurable. Of course, $g = f$ off N. This proves the claim.

However, to prove the result for a simple function, it surely suffices to prove it only for a characteristic function of a measurable set $D \in \mathcal{M}_0$. By definition, D is μ_0-measurable if and only if there are sets $A, B \in \mathcal{B}$ so that $A \subseteq D \subseteq B$ with $\mu(B \setminus A) = 0$. In particular, $\mathbb{I}_A = \mathbb{I}_D$ off of $N = B \setminus A$; this completes the proof.

(c) What does this result say about Lebesgue measurable functions and Borel functions on $\mathbb{R}$? (Compare with problem #14 on page 59 of Rudin.)

ANS: Since the Lebesgue measurable sets (with Lebesgue measure) result from the completion of Lebesgue measure on the Borel sets, we obtain, as a special case, the fact that a Lebesgue measurable function is equal to a Borel function almost everywhere.