Math 103: Measure Theory and Complex Analysis
Fall 2018

10/15/18

Lecture 14

Chapter 2.5 - L^1 spaces

Outline A normed vector space which is complete is called a Banach space. $\mathcal{L}^1(\mu)$ is a vector space, but in general not complete. We can complete $\mathcal{L}^1(\mu)$ to $L^1(\mu)$ by taking equivalence classes of functions where two functions are equivalent if they coincide almost everywhere.

We recall the definition of a norm on a vector space. In the following we assume that $\mathbb{F}$ is a field where $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$.

Definition 1 (Norm) Let V be a vector space over a field $\mathbb{F}$ where $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$. A **norm** is a map

$$\|\cdot\| : V \to [0, +\infty) \text{ such that for all } v, w \in V$$

a) $\|v\| = 0 \Leftrightarrow v = 0$.

b) $\|\lambda \cdot v\| = |\lambda| \cdot \|v\|$ for all $\lambda \in \mathbb{F}$.

c) $\|v + w\| \leq \|v\| + \|w\| \quad (\triangle \neq)$.

Every norm $\|\cdot\|$ induced a metric $d : V \times V \to [0, \infty), (u, v) \mapsto d(u, v) := \|u - v\|$.

Example 2 (p - norms) For $V = \mathbb{C}^n$ or $V = \mathbb{R}^n$ we have for $v = (v_1, v_2, \ldots, v_n) \in V$:

1.) $\|v\|_1 = \sum_{i=1}^n |v_i|$ **(1 - norm)**

2.) $\|v\|_2 = \left(\sum_{i=1}^n |v_i|^2\right)^{\frac{1}{2}}$ **(2 - norm or Euclidean norm)**

3.) $\|v\|_p = \left(\sum_{i=1}^n |v_i|^p\right)^{\frac{1}{p}}$ for $p \in [1, +\infty)$ **(p - norm)**

4.) $\|v\|_\infty = \max_{i \in \{1, \ldots, n\}} |v_i|$ **(∞ - norm)**

Picture Sketch the unit circles in $\mathbb{R}^2$ with respect to 1.), 2.) and 4.):

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Given a measure space $(X, \mathcal{M}, \mu)$ we would like to define a norm on $\mathcal{L}^1(\mu)$ by

$$\boxed{\|f\|_1 = \int_X |f|\, d\mu}.$$

Problem Condition a) of the norm is not satisfied.
Solution We make a vector space out of classes of functions:

Definition 3 (L^1 space) Let $(X, \mathcal{M}, \mu)$ be a measure space. We set

$$L^1(\mu) := \mathcal{L}^1(\mu)/\sim \quad \text{where} \quad f \sim g \Leftrightarrow f = g \text{ almost everywhere.}$$

Theorem 4 Let $(X, \mathcal{M}, \mu)$ be a measure space. Then $(L^1(\mu), \|\cdot\|)$ with the norm

$$\|f\|_1 = \int_X |f|\, d\mu \quad \text{is a normed vector space.}$$

proof 1.) $L^1(\mu)$ is a vector space

We have seen that $\mathcal{L}^1(\mu)$ is a vector space. We can show that

$$W := \{f \in \mathcal{L}^1(\mu) \mid \int_X |f|\, d\mu = 0\}$$

is a subspace. Then it follows from Linear Algebra that the quotient space
$L^1(\mu) := \mathcal{L}^1(\mu)/W$ is again a vector space. We check the subspace criteria:

2.) $\|\cdot\|$ is a norm on $L^1(\mu)$

To show that $\|\cdot\|$ is a norm on $L^1(\mu)$ we remark that $L^1(\mu)$ inherits condition b) and c) of the norm from $\mathcal{L}^1(\mu)$. We recall that c) follows from Minkowski's inequality. So it remains to show that $L^1(\mu) := \mathcal{L}^1(\mu)/W = \mathcal{L}^1(\mu)/\sim$ to complete part a):
Suppose that $\int_X |f|\, d\mu = 0$. We have to show that $f = 0$ almost everywhere. Let

$$E = \{x \in X \mid |f(x)| > 0\}$$

We have to show that $\mu(E) = 0$.

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We prove the statement by contradiction:
Let $E_n = \{x \in X \mid |f(x)| > \frac{1}{n}\}$, then $E = \bigcup_{n\in\mathbb{N}} E_n$. Hence if $\mu(E) > 0$ then

Note Though $L^1(\mu)$ is not a space of functions, we will often pretend that it is.

Remark We recall that a metric space is **complete** if every Cauchy sequence converges in the space. We have:

Definition 6 (Banach space) A normed vector space $(V, \|\cdot\|)$ that is complete is called a **Banach space**.

We want to prove that $(L^1(\mu), \|\cdot\|_1)$ is a Banach space. To this end we prove the following lemma:

Lemma 7 A normed vector space $(V, \|\cdot\|)$ is complete if and only if every absolute convergent series in V is convergent in V.

proof "$\Rightarrow$" We know that every Cauchy sequence is complete. For $(v_k)_{k\in\mathbb{N}}$ let $\sum_{k\in\mathbb{N}} v_k$ be an absolutely convergent series i.e. $\sum_{k\in\mathbb{N}} \|v_k\| = S < \infty$. Let $S_n := \sum_{k=1}^n \|v_k\|$ As $(S_n)_n \subset \mathbb{R}$ is converging to S this implies that $(S_n)_n$ is a Cauchy sequence. Hence

By the $\triangle \neq$ this implies that

Hence the series $(s_n)_n$ where $s_n := \sum_{k=1}^n v_k$ is a Cauchy sequence and $\lim_{n\to\infty} s_n = s \in V$ as $(V, \|\cdot\|)$ is complete. Hence the series converges in V.

"$\Leftarrow$" Now assume that absolute convergence implies convergence of a series and let $(v_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in V. As it is a Cauchy sequence we can extract a subsequence $(v_{n_k})_{k\in\mathbb{N}}$, such that

$$\|v_{n_{k+1}} - v_{n_k}\| < \frac{1}{2^k}. \ (*)$$

We now turn the sequence $(v_{n_k})_{k\in\mathbb{N}}$ into a telescoping sum. To this end we set

Then by (*) we know that $\sum_{k\geq 1}\|a_k\|$ converges, hence by our assumption $\sum_{k\geq 1} a_k$ converges to $v \in V$. This is equal to that the subsequence $(v_{n_k})_k$ converges to v. Then by the $\triangle \neq$ this implies that $(v_n)_{n\in\mathbb{N}}$ converges to $v \in V$.

In total we have shown the lemma. $\square$

Theorem 8 Let $(X, \mathcal{M}, \mu)$ be a measure space then $(L^1(\mu), \|\cdot\|_1)$ is a Banach space.

proof We use the lemma. Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{L}^1(\mu)$ be a sequence of functions, such that $\sum_{n\in\mathbb{N}}\|f_n\|_1 = S \in \mathbb{R}$. We have to show that $\sum_{n\in\mathbb{N}} f_n$ is convergent in $L^1(\mu)$. Let g be the function given by

$$g(x) := \sum_{n\in\mathbb{N}} |f_n(x)| \in [0, +\infty].$$

Then

$$\int_X g\, d\mu =$$

This implies that the set

$$N := \{x \in X \mid g(x) = +\infty\}$$

has measure zero and $\sum_{n\in\mathbb{N}} f_n(x)$ is absolutely convergent for all $x \in X \backslash N$. Let $s(x)$ be the function defined by

$$s(x) = \begin{cases} \sum_{n\in\mathbb{N}} f_n(x) & \text{if} \quad x \notin N \\ 0 & \phantom{\text{if}} \quad x \in N \end{cases}$$

For the function defined by $s_n = \sum_{i=1}^n \mathbb{1}_{X\backslash N} f_i$ we know that

$$|s_n(x)| \leq \qquad\qquad \text{and} \quad \lim_{n\to\infty} s_n(x) =$$

By the DCT we know that

This means that $\sum_{n\in\mathbb{N}} f_n = s$ almost everywhere. $\square$