Math 103: Measure Theory and Complex Analysis
Fall 2018

10/17/18

Lecture 15

Chapter 2.6. - Complex and signed measures

Outline Expanding the definition of a measure, we also allow the measure to have negative values or in $\mathbb{C}$. The first is called a **signed measure**, the second a **complex measure**. By the Jordan decompositon theorem every signed measure can be decomposed into a positive and a negative measure. We will need this fact later to prove the **Radon-Nikodym theorem**.

Definition 1 (complex measure) Let $(X, \mathcal{M})$ be a measurable space. A **complex measure** $\nu : \mathcal{M} \to \mathbb{C}$ is a map, such that

a) $\nu(\emptyset) = 0$.

b) If $(A_i)_{i\in\mathbb{N}} \subset \mathcal{M}$ is a countable union of disjoint sets, then

$$\nu(\biguplus_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} \nu(A_i). \qquad (\sigma \textbf{ addititvity})$$

If $\nu : \mathcal{M} \to \mathbb{R}$ then we call ν a **signed measure**.

Remark 1.) $\nu \neq \pm\infty$ by defintion.
2.) The set $\biguplus_{i\in\mathbb{N}} A_i$ is invariant under rearrangement. Therefore so is the sum $\sum_{i\in\mathbb{N}} \nu(A_i)$. By the **Riemann series theorem** it follows that $\sum_{i\in\mathbb{N}} \nu(A_i)$ converges absolutely.

Definition 2 Let $(X, \mathcal{M})$ be a measurable space and $\nu : \mathcal{M} \to \mathbb{R}$ be a a signed measure. A subset $E \in \mathcal{M}$ is called

a) **positive** if for all $A \in \mathcal{M}$ we have $A \subset E \Rightarrow \nu(A) \geq 0$.

b) **negative** if for all $A \in \mathcal{M}$ we have $A \subset E \Rightarrow \nu(A) \leq 0$.

c) ν **null** if for all $A \in \mathcal{M}$ we have $A \subset E \Rightarrow \nu(A) = 0$.

Remark $\nu(E) = 0$ does not imply that E is ν null.

Picture

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Lemma 3 If $(P_i)_{i\in\mathbb{N}} \subset \mathcal{M}$ are positive sets then $\bigcup_{i\in\mathbb{N}} P_i$ is a positive set.

proof Let $P = \bigcup_{i\in\mathbb{N}} P_i$. We can rewrite P as a disjoint union of sets P_i', where $P_i' \subset P_i$.

Proposition 4 Let $\nu : \mathcal{M} \to \mathbb{R}$ be a signed measure. If $\nu(E) > 0$ then E contains a positive set P with $\nu(P) > 0$.

proof If E is positive, then we are done. If E is not positive, then E contains a measurable set of negative measure. Let

$$\frac{1}{n_1} = \max\left\{\frac{1}{n} \mid n \in \mathbb{N}, \exists E_1 \in \mathcal{M}, E_1 \subset E \text{ and } \boxed{\nu(E_1) \leq -\frac{1}{n}}\right\}$$

For such a set E_1 where $\nu(E_1) \leq -\frac{1}{n_1}$ we have

$$0 < \nu(E) = \qquad\qquad\qquad\qquad \Rightarrow 0 < \nu(E \backslash E_1).$$

If E_1 positive, we are done, otherwise we procced inductively and set for all $k \geq 2$:

$$\frac{1}{n_k} = \max\left\{\frac{1}{n} \mid n \in \mathbb{N}, \exists E_k \in \mathcal{M}, E_k \subset E \backslash \biguplus_{i=1}^{k-1} E_i \text{ and } \boxed{\nu(E_k) \leq -\frac{1}{n}}\right\}$$

Picture

We know that

$$\nu(\biguplus_{i=1}^{k-1} E_i) = \sum_{i=1}^{k-1} \nu(E_i) \leq -\sum_{i=1}^{k-1} \frac{1}{n_i} < 0.$$

Hence

$$\Rightarrow 0 < \nu(E \backslash \biguplus_{i=1}^{k-1} E_i).$$

Now if for some k we have that $E \backslash \biguplus_{i=1}^{k-1} E_i \subset E$ is positive the above inequality implies that our statement is true and we are done.
If the process does not end, then we set $\boxed{A = E \backslash \biguplus_{i \in \mathbb{N}} E_i}$. Then we have again, as above that $\boxed{0 < \nu(A)}$. On the other hand, as $\biguplus_{i \in \mathbb{N}} E_i \in \mathcal{M}$ and ν only takes finite values

$$-\infty < \nu(\biguplus_{i \in \mathbb{N}} E_i) =$$

That means that

Now fix $\epsilon > 0$. Then there is $\frac{1}{n_{k-1}}$, such that $\frac{1}{n_{k-1}} < \epsilon$ (as $\lim_{k \to \infty} \frac{1}{n_k} = 0$). Furthermore $A \subset E \backslash \biguplus_{i=1}^{k-1} E_i$. By the maximality of $\frac{1}{n_{k-1}}$ we know that A contains no measurable set F with $\nu(F) \leq -\frac{1}{n_{k-1}}$. In other words for all $F \subset A, F \in \mathcal{M}$ we have that

$$\nu(F) > -\frac{1}{n_{k-1}} > -\epsilon.$$

As this is true for all $\epsilon > 0$ this implies that A is positive. Hence again we have found a set that satisfies our condtions. This concludes the proof of **Proposition 4** $\square$

Proposition 5 (Hahn decomposition) Let $(X, \mathcal{M})$ be a measure space and $\nu : \mathcal{M} \to \mathbb{R}$ be a signed measure. Then there is a partition

$$X = P \uplus N \text{ where } P \text{ is positive and } N \text{ is negative.}$$

proof Let $\mathcal{P}$ be the collection of positive sets in X. We set

$$\lambda = \sup\{\nu(P) \mid P \in \mathcal{P}\} \in [0, \infty]$$

Then there is a sequence $(P_i)_{i \in \mathbb{N}} \subset \mathcal{P}$ such that $\lim_{i \to \infty} \nu(P_i) = \lambda$. Take $P = \bigcup_{i \in \mathbb{N}} P_i$. We show that $\nu(P) = \lambda$.

We now show that P is "the **largest**" positive subset.
Now set $N = X \backslash P$ and suppose that there is an $E \subset N$, such that E is positive. Then $P \uplus E$ is positive by the lemma and so by the definition of λ

Now if N would contain a subset $E' \in \mathcal{M}$ with positive measure $\nu(E') > 0$, then

This means that N is a negative set and our decomposition follows. $\square$

Definition 6 We call $\{P, N\}$ the **Hahn decomposition** of X. It is unique up to null sets.

Definition 7 Let $(X, \mathcal{M})$ be a measure space. Two positive measures μ_1 and μ_2 are said to be **mutually singular** if there is a partition of X

$$X = X_1 \uplus X_2 \quad \text{such that} \quad \mu_1(X_2) = \mu_2(X_1) = 0.$$

In this case we write shortly $\mu_1 \perp \mu_2$.

Theorem 8 (Jordan decomposition) Let $(X, \mathcal{M})$ be a measure space and $\nu : \mathcal{M} \to \mathbb{R}$ be a signed measure. Then there is a unique (up to sets of measure zero) pair (ν^+, ν^-) of mutually singular positive measures, such that $\nu = \nu^+ - \nu^-$.

proof Let $\{P, N\}$ be the Hahn decomposition of X, i.e.

$$X = P \uplus N \quad \text{where} \quad P \text{ is positive and } N \text{ is negative.}$$

We set for all $E \in \mathcal{M}$:

$$\nu^+(E) = \nu(E \cap P) \quad \text{and} \quad \nu^-(E) = -\nu(E \cap N)$$

The rest is an exercise. $\square$