Math 103: Measure Theory and Complex Analysis
Fall 2018

10/29/18

Lecture 20

Reminder $e^{x+iy} \overset{Def.}{=} e^x \cdot (\cos(y) + i\sin(y))$. Then

$$f(z) = e^z \in \mathcal{H}(\mathbb{C}) \quad \text{and} \quad f' = f.$$

Picture Plot e^z using the grid map i.e. look at $\{f(x+iy), x = const.\}$ and $\{f(x+iy), y = const.\}$

Furthermore $\boxed{e^w = 1 \Leftrightarrow w = 2\pi \cdot i \cdot k}$ for some $k \in \mathbb{Z}$.
Given the picture above how can we define the inverse function log?

Remark 8 For any continuous $f : \Omega \to \mathbb{C}$ we have

$$\left| \int_\gamma f(z)\, dz \right| \leq \int_\gamma |f(z)|\, dz \leq M\ell(\gamma) \quad \text{where} \quad M = \max\{|f(z)|, z \in \gamma^*\}.$$

proof

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Theorem 9 Let γ be a closed path and $\Omega = \mathbb{C} \setminus \gamma^*$. If $z \in \Omega$, define

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2i\pi} \cdot \int_\gamma \frac{1}{w - z} \, dw$$

Then $\mathrm{Ind}_\gamma : \Omega \to \mathbb{Z}$ is constant on connected components of Ω and 0 on the unbounded component. $\mathrm{Ind}_\gamma(z)$ is called the **winding number** of γ around z.

Picture

proof Note that $\gamma^\star$ is compact. Hence $\exists R > 0$ such that $\gamma^\star \subset D_R(0)$. But $D_R(0)^c$ is connected and must lie in a single component of Ω. Since all of the components live inside of $D_R(0)$, there is a unique unbounded component.

Now fix $z \in \Omega$. Then

$$2\pi i \ \mathrm{Ind}_\gamma(z) = \int_a^b \frac{\gamma'(t)}{\gamma(t) - z} \, dt.$$

Let

$$\varphi(t) = \exp\left(\int_a^t \frac{\gamma'(t)}{\gamma(t) - z} \, dt \right).$$

We have: $\boxed{\mathrm{Ind}_\gamma(z) \in \mathbb{Z} \Longleftrightarrow \varphi(b) = 1.}$ and $\varphi(t) \neq 0$ for all t:

Furthermore by the chain rule

$$\varphi'(t) =$$

This implies that

$$\frac{\varphi'(t)}{\varphi(t)} =$$

except for possibly a finite set $F \subseteq [a, b]$ where γ is not differentiable.

Thus

$$\varphi'(t)\big(\gamma(t)-z\big)-\gamma'(t)\varphi(t)=0$$

for $t \in [a,b] \setminus F$. Therefore let,

$$g(t)=\frac{\varphi(t)}{\gamma(t)-z}$$

Then for $g(t)$ we have:

$$g'(t)=$$

Hence $g(t)$ is continuous on $[a,b]$ and $g'(t)=$ for $t \in [a,b] \setminus F$.
So g is

Since $\boxed{\varphi(a)=}$,

$$g(t)= \qquad \forall t \in [a,b]$$

$\implies \varphi(t)=\frac{\gamma(t)-z}{\gamma(a)-z}$ and since $\gamma(a)=\gamma(b)$, $\boxed{\varphi(b)=1}$. $\square$

By **Ch. 1 Theorem 12** applied to $X=[0,1]$, $\varphi=\gamma, \nu=\gamma'\cdot\lambda$ we know that Ind_γ is analytic, hence continuous.

Picture

Since it is integer-valued, it must be constant on connected components. Finally, if $\gamma^\star \subset D_R(0)$ and $|z| \geq R$,

$$|\mathrm{Ind}_\gamma(z)| \leq$$

So $|\mathrm{Ind}_\gamma(z)|<1$ for z large enough, hence $\mathrm{Ind}_\gamma(z)=0$. $\square$

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Theorem 10 Let γ be a positively oriented circle with radius $r > 0$ centered at a: $\gamma(t) = e^{it} + a$ for $t \in [0, 2\pi]$. Then

$$\text{Ind}_\gamma(a) = \begin{cases} 1 & \text{if } |z-a| < r \\ 0 & \text{if } |z-a| > r. \end{cases}$$

Picture

proof $\Omega = \mathbb{C} \setminus \gamma^\star$ has two components and $|z-a| > r$ is the unbounded one.

$$\text{Ind}_\gamma(a) =$$

Theorem 11 (Fundamental Theorem for line integrals) Suppose $F \in \mathcal{H}(\Omega)$ and F' is continuous on Ω. If $\gamma : [a, b] \to \Omega$ is a path, then

$$\int_\gamma F'(z)\,dz = F\big(\gamma(b)\big) - F\big(\gamma(a)\big).$$

In particular,

$$\int_\gamma F'(z)\,dz = 0 \qquad \text{for all closed path in } \Omega.$$

proof

$$\int_\gamma F'(z)\,dz = \qquad \square$$

Corollary 12 If γ be a closed path in Ω, then

$$(\star) \qquad \int_\gamma z^n\,dz = 0 \quad \text{for} \quad n \in \mathbb{N}.$$
