Math 103: Measure Theory and Complex Analysis
Fall 2018

11/5/18

Lecture 23

Outline We investigate the behavior of $f \in \mathcal{H}(\Omega \backslash \{a\})$ in $z = a$.

Corollary 12 The zeros of non-constant analytic functions are isolated.

Corollary If f is holomorphic in a region Ω, then $Z(f)$ is at most countable (if $f \not\equiv 0$).

proof: . □

Corollary 13 (holomorphic extension) Suppose $f, g \in \mathcal{H}(\Omega)$ and $\{z : f(z) = g(z)\}$ has a limit point in Ω. Then $f = g$

proof: . □

Example The function

$$\exp(z) = \sum_{n \geq 0} \frac{z^n}{n!}$$

is the *only* entire function extending the real function $x \mapsto e^x$.

Example Let $f(z) = \sin\left(\frac{1}{z}\right)$ on $\Omega = D_1'(0)$ with

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

Then $f\left(\frac{1}{k\pi}\right) = 0$ for all $k \in \mathbb{Z}_+$, but $f \not\equiv 0$ because $0 \notin \Omega$.

Picture Sketch $|f|$ near 0.

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Note (Behavior if $f'(a) = 0$) Let $f \in \mathcal{H}(\Omega)$ and $f'(a) = 0$ for some $a \in \mathbb{C}$. By the power series expansion, we have in some disk $D_R(a)$

$$f(z) = f(a) + c_m \cdot (z-a)^m + \sum_{k>m} c_k (z-a)^k.$$

We look at the vectors $u = a + r \cdot e^{i\varphi}$ and $v = a + r \cdot e^{i\psi}$. $\angle(u, v) = |\varphi - \psi| = \angle(\frac{u-a}{v-a}, 1)$

Picture

We may assume that $a = 0$. Looking at the image we get for some $m \geq 2$:

$$f(u) = f(0) + c_m \cdot u^m + \sum_{k>m} c_k u^k \quad \text{and} \quad f(v) = f(0) + c_m \cdot v^m + \sum_{k>m} c_k v^k.$$

$$\frac{f(u)-f(0)}{f(v)-f(0)} = \qquad \text{and}$$

$$\lim_{r \to 0} \frac{f(r \cdot e^{i\varphi}) - f(0)}{f(r \cdot e^{i\psi}) - f(0)} =$$

Hence for the angle we get $\angle(f(u), f(v)) =$.

Chapter 4 - Singularities

Definition 1 If $a \in \Omega$ and $f \in \mathcal{H}(\Omega \setminus \{a\})$, then a is called an **isolated singularity** of f.

If a is an isolated singularity of f in Ω and $f(a)$ can be (re)defined to make $f \in \mathcal{H}(\Omega)$, then a is called a **removable singularity**.

Example Let

$$\sin(z) = z - \frac{z^3}{3!} + \cdots = \sum_{n \geq 0} \frac{z^{2n+1}}{(2n+1)!}(-1)^n$$

Then $f(z) = \frac{\sin(z)}{z} \in \mathcal{H}(\mathbb{C}\setminus\{0\})$. For all $z \neq 0$,

$$f(z) =$$

We can define $f(0) =$.

Remark 2 If f is continuous on Ω and $f \in \mathcal{H}(\Omega\setminus\{a\})$, then it follows from **Morera's Theorem** and **Cauchy's Theorem for** $\triangle$ that $f \in \mathcal{H}(\Omega)$, so that a is a removable singularity (with $f(a)$ as-is).

Theorem 3 If $f \in \mathcal{H}(\Omega\setminus\{a\})$ and $\exists r > 0$ such that $D_r(a) \subset \Omega$ and $|f(z)|$ is bounded on $D'_r(a)$, then f has a removable singularity at a.

proof Let

$$h(z) = \begin{cases} (z-a)^2 f(z) & \text{if } z \neq a \\ 0 & \text{if } z = a \end{cases}$$

Since f is bounded near a, we have that $h'(a) = 0$:

so $h \in \mathcal{H}(\Omega)$. Thus $\exists \{c_n\}_{n\geq 2}$ such that $h(z) = \sum_{n\geq 2} c_n(z-a)^n$ for $z \in D_r(a)$.
If $z \neq a$, then

$$h(z) = (z-a)^2 f(z) \implies$$

Thus if we set $f(a) = c_2$, we see $f \in \mathcal{H}(D_r(a))$. $\square$

Theorem 4 (Classification of Isolated Singularities)
Suppose f has an isolated singularity at $a \in \Omega$ and $D_r(a) \subset \Omega$. Then, exactly one of the following holds:

a) a is a removable singularity

b) $\exists b_1, \ldots, b_m \in \mathbb{C}$ such that $b_m \neq 0$ and

$$f(z) - \sum_{j=1}^{m} \frac{b_j}{(z-a)^j}$$

has a removable singularity at a. In this case, we say that f has a **pole of order m** at a.

c) For all $0 < r' \leq r$, $f(D'_{r'}(a))$ is dense in $\mathbb{C}$. In this case, we say that f has an **essential singularity** at a.

proof Suppose c) fails. Then $\exists \delta > 0$ and $w \in \mathbb{C}$ such that

$$\forall z \in D'_{r'}(a),\ |f(z) - w| > \delta$$

Let $\boxed{g(z) = \dfrac{1}{f(z) - w}}$. Then $g \in \mathcal{H}(D'_{r'}(a))$ and $|g(z)| \leq \frac{1}{\delta}$

Thus g has a removable singularity at a and we can define $g(a)$ such that $g \in \mathcal{H}(D_{r'}(a))$.

Case I $g(a) \neq 0$.
Then $f(z) = \frac{1}{g(z)} + w$ near a and f is bounded in some $D'_\rho(a)$ with $0 < \rho < r'$. Hence f has a removable singularity at a.

Case II $g(a) = 0$.
Clearly, a is an isolated zero of g, hence $\exists m \in \mathbb{Z}_+$ such that

$$g(z) = (z - a)^m h(z)$$

with $h \in \mathcal{H}(D_r(a))$ and $h(a) \neq 0$.
But $\exists 0 < \rho < r$ such that $h(z) \neq 0$ in $D_\rho(a)$.

$$\implies \frac{1}{f(z) - w} = \qquad\qquad \text{for} \quad z \in D'_\rho(a)$$

$$\implies f(z) = \qquad\qquad \text{for} \quad z \in D'_\rho(a)$$

Then

$$f(z) - \sum_{n=0}^{m-1} c_n (z - a)^{n-m} =$$

Where the last expression is analytic in $D_\rho(a)$

Hence $f(z) - \sum_{n=0}^{m-1} c_n (z - a)^{n-m}$ has a removable singularity at a and $c_0 = \frac{1}{h(a)} \neq 0$. $\square$