Math 103: Measure Theory and Complex Analysis
Fall 2018

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Lecture 24

Corollary 5 If $f \in \mathcal{H}(\Omega \setminus \{a\})$, then

$$f \text{ has a pole at a } \iff \lim_{z\to a} |f(z)| = +\infty$$

More precisely, f has a pole of order m at a iff:

$$\lim_{z\to a} |z-a|^{m-1} |f(z)| = \infty \quad \text{and} \quad \lim_{z\to a} |z-a|^{m} |f(z)| = L < \infty$$

proof Exercise.

Proposition 6 (Cauchy's Estimate) Suppose $f \in \mathcal{H}(\Omega)$ and $D_R(a) \subseteq \Omega$. If $|f(z)| \leq M$ for $z \in D_R(a)$, then $\left|f^{(n)}(a)\right| \leq \frac{n!M}{R^n}$

proof Let $0 < r < R$ and $\gamma(t) = a + re^{it}$ for $t \in (0, 2\pi]$ Since $D_R(a)$ is convex and $\mathrm{Ind}_\gamma(a) = 1$, we know

$$f^{(n)}(a) = \frac{n!}{2i\pi} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} dw$$

Thus

$$\left|f^{(n)}(a)\right| \leq$$

and this holds for every $r < R$. □

Picture

Liouville's Theorem A bounded entire function f i.e. $f \in \mathcal{H}(\mathbb{C})$ is constant.

proof Suppose $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then for $a = 0$ in **Prop. 6** we get

Hence f is constant. □

Definition 8 A sequence $\{f_n\}_n$ of functions on Ω is said to **converge** to $f : \Omega \to \mathbb{C}$ **uniformly on compact subsets** of Ω if:

$\forall \varepsilon > 0$, $\forall$ compact $K \subset \Omega$, $\exists N = N(\varepsilon, K)$ such that $z \in K$ and $n \geq N \implies |f_n(z) - f(z)| < \varepsilon$

Picture

Example If $f(z) = \sum_{n\geq 0} c_n(z-a)^n$ for $z \in D_r(a)$, then the RHS converges uniformly on compact subsets. (Exercise)

Example ($\mathbb{R}$ world): Let $f_n(x) = \frac{\sin(\pi n x)}{\sqrt{n}}$ for $x \in [0,1]$. Then $f_n \to 0$ uniformly on $[0,1]$. But, $f_n'(x) =$ and $f_n' \nrightarrow 0$, not even point-wise.
(Notice that $f_n'(x)$ does not converge for any $x \in [0,1]$)

Picture

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In the complex world...

Theorem 9 Suppose $\{f_n\}_n \subset \mathcal{H}(\Omega)$ and $f_n \to f$ uniformly on compacts. Then $f \in \mathcal{H}(\Omega)$ and $f_n' \to f'$ uniformly on on compacts.

proof 1.) f **is continuous in** Ω**:** The convergence is uniform on all closed disks. Take $a \in \Omega$ and $D_\delta(a)$. We use that f_n is continuous for all $n \in \mathbb{N}$ and the uniform convergence on $D_\delta(a)$. Then we use the $\triangle \neq$ with a 3 ϵ proof:

2.) f **is holomorphic in** Ω**:** We use **Morera's theorem**. Note that if D is a disk in Ω and γ is a closed path in D, then γ^* is compact and $\displaystyle\int_\gamma f(z)dz = \lim_{n\to\infty}\int_\gamma f_n(z)dz = 0$.
Picture

proof:

Thus $f \in \mathcal{H}(D)$ by **Morera's Theorem**, so $f \in \mathcal{H}(\Omega)$.

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3.) $f_n' \to f'$ **uniformly on compacts:** Let $K \subset \Omega$ be compact.

Claim: $\exists K' \subset \Omega$ and $r > 0$ such that

a) K' is compact

b) $K \subset K' \subset \Omega$ and $\forall z \in K$, $\overline{D_r(z)} \subset K'$

Picture

Assume that the claim is true and set

$$M_n = \sup\{|f_n(z) - f(z)|, z \in K'\} \geq \sup\{|f_n(z) - f(z)|, |z - w| < r, w \in K\}$$

By the **Cauchy estimate**, if $w \in K$ and $D_r(w) \subset K'$,

$$\left|f_n'(w) - f'(w)\right| \leq$$

But K' is compact, so $M_n \to 0$. Since K', M_n and r only depend on K, $f_n' \xrightarrow{uniformly} f'$ on K.

proof of Claim As K compact there $\exists \delta > 0$ such that $z \in K \Rightarrow D_{2\delta}(z) \subset \Omega$. Since K is compact, $\exists z_1, \ldots, z_n$ such that

$$K \subseteq \bigcup_{j=1}^{n} D_\delta(z_j)$$

Let $K' = \bigcup_{j=1}^{n} \overline{D_\delta(z_j)} \subset \bigcup_{j=1}^{n} D_{2\delta}(z_j) \subset \Omega$.

Now we can find $r > 0$ such that $z \in K \Rightarrow D_r(z) \subset K'$. ☐

Corollary 10 All the derivatives of the f_n converge uniformly on compacts:

$$f_n^{(k)} \xrightarrow{uniformly} f^{(k)} \quad \text{for all} \quad k \geq 0.$$

Note Remember that on $\mathbb{R}$, sequences of smooth functions can converge to nowhere differentiable functions.
