Math 103: Measure Theory and Complex Analysis
Fall 2018

09/17/18

Lecture 3

Aim: Create a good theory of measure and measurable sets.

Proposition 15 Suppose that $u, v : (X, \mathcal{M}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are measurable and that $\Phi : \mathbb{R}^2 \to Y$ is continuous. Then the function

$$h : (X, \mathcal{M}) \to (Y, \mathcal{B}(Y)), x \mapsto h(x) = \Phi(u(x), v(x)) \text{ is measurable.}$$

Picture

proof It is sufficient to show that the map

$$f : (X, \mathcal{M}) \to (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)), x \mapsto f(x) = (u(x), v(x))$$

is measurable. Then $h = \Phi \circ f$ is a composition of measurable functions and therefore measurable. To see that f is measurable we recall that the open rectangles with endpoints in $\mathbb{Q}^2$ form a countable basis β of the topology of $\mathbb{R}^2$. By **Lemma 14** it is therefore sufficient to show that for all $R = (a_1, a_2) \times (b_1, b_2) \in \beta$ we have that $f^{-1}(R) \in \mathcal{M}$. But

In total f is a measurable function which implies that $h = \Phi \circ f$ is measurable. $\square$

Corollary 16 Suppose that $f, g : (X, \mathcal{M}) \to \mathbb{R}$ are measurable. Then

$$f \pm g \ , \ f \cdot g \text{ and } f + i \cdot g \text{ are measurable.}$$

proof (With $u = f, v = g$) we take the continuous functions

Then the corollary follows from our proposition. □

Corollary 17 Suppose that $f, g : (X, \mathcal{M}) \to \mathbb{C}$ are measurable. Then

$$|f| \ , \ \mathrm{Re}(f) \ , \ \mathrm{Im}(f) \text{ and } f \pm g \ , \ f \cdot g \text{ are measurable.}$$

proof Idea: These are consequences of **Remark 11** and **Proposition 15**:
For $|f|, \mathrm{Re}(f)$ and $\mathrm{Im}(f)$ we note that

$$z \to |z| \, , \, z \to \mathrm{Re}(z) \ \text{ and } \ z \to \mathrm{Im}(z)$$

are continuous functions. Hence $|\cdot| \circ f, \mathrm{Re} \circ f$ and $\mathrm{Im} \circ f$ are each the composition of a measurable with a continuous function. Hence these composition are measurable by **Remark 11**.

We prove the statement for $f - g$ and $f \cdot g$ in a similar fashion. □

Chapter 1.3 The extended real line

Aim: We want to allow real valued functions to take the values $+\infty$ or $-\infty$ so if $(f_n))_{n \in \mathbb{N}}$ is a sequence of functions we can consider

$$f(x) := \sup_{n \in \mathbb{N}} f_n(x)$$

without fussing.

Definition 1 The **extended real line** $\bar{\mathbb{R}} = [-\infty, \infty]$ is the topological space $\mathbb{R} \cup \{\pm\infty\}$ with the topology $\mathcal{T}$ whose basis $\bar{\beta}$ are the sets

$$(a, b)\ ,\ [-\infty, a) = \{-\infty\} \cup (-\infty, a)\ ,\ (b, +\infty) \cup \{+\infty\} = (b, +\infty], \text{ where } a, b \in \mathbb{R}\,.$$

Remark By taking $a, b \in \mathbb{Q}$ in the above definition, we see that $\bar{\mathbb{R}}$ is second countable.

Lemma 2 A function $f : (X, \mathcal{M}) \to \bar{\mathbb{R}}$ is measurable, if and only if

$$f^{-1}((a, +\infty]) \in \mathcal{M} \text{ for all } a \in \mathbb{R}\,.$$

proof "$\Rightarrow$" Clearly, as f is measurable and the set $(a, +\infty] \subset \mathcal{T}$ for all $a \in \mathbb{R}$ we know that $f^{-1}((a, +\infty]) \in \mathcal{M}$ for all $a \in \mathbb{R}$.
"$\Leftarrow$" We have to show that $f^{-1}(B) \in \mathcal{M}$ for any element B of the basis $\bar{\beta}$. Then by **Lemma 14** and as $\bar{\mathbb{R}}$ is second countable, we know that f is measurable.

We first prove this for the sets of the form $[-\infty, a]$:

Then we prove it for open intervals $[-\infty, a)$ using countable unions. We know that $[-\infty, b) =$

Finally, $f^{-1}((a, b)) = f^{-1}([-\infty, b)) \cap f^{-1}((a, +\infty)) \in \mathcal{M}$.
Hence our statement is true. □

lim inf and lim sup

We recall the following definitions from real analysis:

Let $(a_n)_{n\in\mathbb{N}} \subset \bar{\mathbb{R}}$ be a sequence. For $k \geq 1$ consider the new sequence

$$b_k = \sup_{n \geq k} a_n = \sup\{a_k, a_{k+1}, a_{k+2}, a_{k+3}, \dots\}$$

Then $b_{k+1} \leq b_k$ for all $k \in \mathbb{N}$ and therefore $\lim_k b_k = \inf_{k\in\mathbb{N}} b_k \in \bar{\mathbb{R}}$. We define

$$\limsup_{n\in\mathbb{N}} a_n \overset{\textbf{Def.}}{=} \lim_{k\to\infty} b_k = \inf_{k\in\mathbb{N}} b_k.$$

In a similar fashion we define

$$\liminf_{n\in\mathbb{N}} a_n \overset{\textbf{Def.}}{=} \lim_{k\to\infty} \inf_{k\geq n} a_n.$$

Example Sketch the sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n := \frac{\cos(n)}{n}$. Then sketch the sequences $(\sup_{n\geq k} a_n)_k$ and $(\inf_{n\geq k} a_n)_k$.

Proposition 3 For a sequence $(a_n)_{n\in\mathbb{N}} \subset \bar{\mathbb{R}}$ we have that

a) $\liminf_{n\in\mathbb{N}} a_n \leq \limsup_{n\in\mathbb{N}} a_n$.

b) $\lim_{n\to\infty} a_n$ exists if and only if $\liminf_{n\in\mathbb{N}} a_n = \lim_{n\to\infty} a_n = \limsup_{n\in\mathbb{N}} a_n$.

proof Look it up.

Theorem 4 Suppose that $f_n : (X, \mathcal{M}) \to \bar{\mathbb{R}}$ is measurable for all $n \in \mathbb{N}$ Then so are

$$g = \sup_{n\in\mathbb{N}} f_n \ , \ h = \limsup_{n\in\mathbb{N}} f_n \ , \ p = \inf_{n\in\mathbb{N}} f_n \ \text{ and } \ q = \limsup_{n\in\mathbb{N}} f_n. \text{ (pointwise)}$$

proof Idea: We use **Lemma 2**:

1.) g : We have to show that for all $a \in \mathbb{R}$
we have that $g^{-1}((a, +\infty]) = \{x \in X \mid g(x) > a\} \in \mathcal{M}$. But

Hence $g^{-1}((a, +\infty]) = \bigcup_{n\in\mathbb{N}} f_n^{-1}((a, +\infty]) \in \mathcal{M}$. Hence g is a measurable function.

2.) p :

3.) h : Using 1.) and 2,) we see that

4.) q : In a similar fashion, as $q = \sup_{k\in\mathbb{N}}(\inf_{n\geq k} f_n)$ is measurable. $\square$

Math 103: Measure Theory and Complex Analysis
Fall 2018

09/17/18

This theorem implies that **measurability is preserved under pointwise limits**.

Corollary 5 Suppose $Y = \bar{\mathbb{R}}$ or $Y = \mathbb{C}$ and let $f_n : (X, \mathcal{M}) \to Y$ be measurable for all $n \in \mathbb{N}$. Then if $f_n(x) \overset{n\to\infty}{\to} f(x)$ for all $x \in X$ then $f : (X, \mathcal{M}) \to Y$ is measurable.

proof If $Y = \bar{\mathbb{R}}$ we know that $\lim_{n\to\infty} f_n(x) = \limsup_{n\in\mathbb{N}} f_n(x)$. Hence the result follows from the theorem. If $Y = \mathbb{C}$ then

□
