Math 103: Measure Theory and Complex Analysis
Fall 2018

09/19/18

Lecture 4

Chapter 1.4 Simple functions

Aim: A function is Riemann integrable if it can be approximated with step functions. A function is Lebesque integrable if it can be approximated with measurable **simple functions**. A measurable simple function is similar to a step function, just that the supporting sets are elements of a σ algebra.

Picture

Definition 1 (Simple functions) A function $s : X \to \mathbb{C}$ is called a **simple function** if it has finite range. We say that s is a **non-negative** simple function (nnsf) if $s(X) \subset [0, +\infty)$.

Note 2 If $s(X) \neq \{0\}$, then $s(X) = \{a_1, a_2, a_3, \dots, a_n\}$ and let $A_i = \{x \in X \mid s(x) = a_i\}$. Then s is measurable if and only if $A_i \in \mathcal{M}$ for all $i \in \{1, 2, \dots, n\}$. In this case we have that

$$s = \sum_{i=1}^{n} a_i \cdot \mathbb{1}_{A_i}. \tag{1}$$

We can furthermore assume that the $(A_i)_i$ are mutually disjoint. This representation as a linear combination of characteristic functions is unique, if the $(a_i)_i$ are distinct and non-zero. In this case we call it the **standard representation of** s.

Theorem 3 (Approximation by simple functions) For any function $f : (X, \mathcal{M}) \to [0, +\infty]$, there are nnsfs $(s_n)_{n\in\mathbb{N}}$ on X, such that

a) $0 \leq s_1 \leq s_2 \leq \ldots \leq f$.

b) For all $x \in X$, we have $\lim_{n\to\infty} s_n(x) = f(x)$.

Furthermore, if f is measurable, then the $(s_n)_n$ can be chosen measurable as well.

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proof Let $\varphi_n : [0, +\infty] \to \mathbb{R}$ be the function defined in the following way: let $k_n(x) = k$ be the unique integer, such that

$$k \cdot 2^{-n} \leq x \leq (k+1) \cdot 2^{-n} \text{ and set } \varphi_n(x) := \begin{cases} k \cdot 2^{-n} & \text{if} \quad 0 \leq x < n \\ n & \quad\;\; n \leq x \leq \infty \end{cases}. \tag{2}$$

Example Sktech φ_2 and φ_3.

Then $\varphi_n : [0, +\infty] \to \mathbb{R}$ is a Borel map and

$$0 \leq \varphi_1(x) \leq \varphi_2(x) \leq \varphi_3(x) \ldots \leq x \text{ for all } x \in \mathbb{R}.$$

In fact, if $x \in [0, n]$ then by the definition of $k_n(x) = k$ and φ_n in (2) we have that

$$x - 2^{-n} \leq \varphi_n(x) \leq x \text{ hence } \lim_{n \to \infty} \varphi_n = id.$$

We now set $s_n = \varphi_n \circ f$.
Write out $s_n = \varphi_n(f(x))$ and sketch s_5 for $f(x) := x^2$.

Since φ_n is Borel, we know that s_n is measurable if f is measurable and a) and b) are easily verified. □

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Chapter 1.5 Measures

Aim: By given each element of a σ algebra a weight, we can define a measure. All we need is that the measure is countably additive. This is the last step to define integration.

Definition 1 (Measure) Let $(X, \mathcal{M})$ be a measurable space.

a) A function $\mu : \mathcal{M} \to [0, +\infty] \subset \bar{\mathbb{R}}$ is called a **positive measure** if it is **countably additive**, i.e. if $(A_i)_{i\in\mathbb{N}}$ is a collection of mutually disjoint elements of $\mathcal{M}$, then

$$\mu\left(\biguplus_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i)$$

To avoid trivialities, we also assume that there is an $A \in \mathcal{M}$, such that $\boxed{\mu(A) < \infty}$.

b) A space $(X, \mathcal{M}, \mu)$ is called a **measure space**.

c) A function $\mu : \mathcal{M} \to \mathbb{C}$ that is countably additive is called a **complex measure**.

Note If not mentioned otherwise we assume in the following that a measure is a positive measure.

Theorem 2 (Properties of μ) Let $(X, \mathcal{M}, \mu)$ be a measure space. Then

a) $\mu(\emptyset) = 0$.

b) If $A \subset B$ then $\mu(A) \leq \mu(B)$.

c) If $(A_n)_{n\in\mathbb{N}} \subset \mathcal{M}$ and

$$A_1 \subset A_2 \subset A_3 \subset \ldots \quad \text{and} \quad A = \bigcup_{n\in\mathbb{N}} A_n$$

then $\lim_{n\to\infty} \mu(A_n) = \mu(A)$.

d) If $(A_n)_{n\in\mathbb{N}} \subset \mathcal{M}$ and $\boxed{\mu(A_1) < \infty}$. If furthermore

$$A_1 \supset A_2 \supset A_3 \supset \ldots \quad \text{and} \quad A = \bigcap_{n\in\mathbb{N}} A_n$$

then $\lim_{n\to\infty} \mu(A_n) = \mu(A)$.

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proof

a) $\mu(\emptyset) = 0$: Take $A \in \mathcal{M}$, such that $\mu(A) < \infty$. Then

b) If $A \subset B$ then $\mu(A) \leq \mu(B)$:

c) Idea: We divide A into a telescoping sum of sets:

d) Idea: We divide A_1 into a telescoping sum of sets:

In total this settles the proof of the theorem. □

Examples We give a few simple examples. Let X be a set and $\mathcal{M} = \mathcal{P}(X)$.

a) **Counting measure:** For any $E \subset X$ we set

$$\mu(E) = \begin{cases} +\infty & \text{if} \quad \#(E) = +\infty \\ n & \phantom{\text{if}} \quad \#(E) = n \end{cases} .$$

b) **Unit mass concentrated at x_0:** For fixed $x_0 \in X$ we set

$$\mu(E) = \begin{cases} 1 & \text{if} \quad x_0 \in E \\ 0 & \phantom{\text{if}} \quad x_0 \notin E \end{cases} .$$

c) The hypothesis that $\mu(A_1) < \infty$ can not be dropped in **Theorem 2**, part d): For $X = \mathbb{N}$ let μ be the counting measure. Let $A_n = \{k \in \mathbb{N} \mid k \geq n\}$. Then

$$A = \bigcap_{n=1}^{\infty} A_n = \emptyset \Rightarrow \mu(A) = 0 \;\text{ but }\; \mu(A_n) = \infty \text{ for all } n \in \mathbb{N} .$$

Hence $\lim_{n\to\infty} \mu(A_n) \neq \mu(A)$.