# Math 103: Measure Theory and Complex Analysis
# Fall 2018

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**Lecture 5**

## Chapter 1.6 Integration

**Outline:** A function is Riemann integrable if it can be "approximated" by step functions. A function is Lebesque integrable if it can be "approximated" by simple functions.
In the following section we we assume that $(X, \mathcal{M}, \mu)$ is a measure space and $\mu$ is a positive measure. We start with the definition of integration for simple functions.

**Prelude: Arithmetic in $\bar{\mathbb{R}}$**

**Outline** To define integration properly we have to deal with functions that take values in $\{\pm\infty\}$. To make this work we have to set a few conventions.

**Definition 1** On $[0, \infty] = [0, +\infty] =\subset \bar{\mathbb{R}}$ we define:

a) Addition: $a + \infty = \infty + a = \infty$ if $0 \leq a \leq \infty$.

b) Multiplication:

$$a \cdot \infty = \infty \cdot a = \begin{cases} \infty & \text{if} \quad 0 < a \leq \infty \\ 0 & \phantom{\text{if}} \quad a = 0 \end{cases}.$$

**Note** One verifies that with this definition in $([0, \infty], +, \cdot)$

- $+$ and $\cdot$ are commutative and associative operations.
- In $([0, \infty], +, \cdot)$ the distributive laws hold.
- $a + b = a + c \Rightarrow b = c$ for $a < \infty$
  $a \cdot b = a \cdot c \Rightarrow b = c$ for $0 < a < \infty$
- If $(a_n)_n$ and $(b_n)_n$ are increasing sequences in $[0, \infty]$, such that

$$\lim_{n\to\infty} a_n = a \quad \text{and} \quad \lim_{n\to\infty} b_n = b \quad \text{then} \quad \lim_{n\to\infty} a_n \cdot b_n = a \cdot b.$$

The last statement together with **Ch.1.3 Theorem 4** and **Ch.1.4 Theorem 3** implies that

**Theorem 2** $f, g : (X, \mathcal{M}) \to [0, \infty]$ measurable then

$$f + g \quad \text{and} \quad f \cdot g \text{ measurable.}$$

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**Picture** Sketch the function $3 \cdot \mathbb{1}_A$ and the function $2 \cdot \mathbb{1}_B$ and $3 \cdot \mathbb{1}_A + 2 \cdot \mathbb{1}_B$ for some $A, B \subset \mathbb{R}$.

**proof of Theorem 2**

$\square$

**Integration of simple functions**

**Definition 3 (Integration)** Let $s : X \to [0, \infty)$ be measurable simple function in the form

$$s = \sum_{i=1}^{n} a_i \cdot \mathbb{1}_{A_i} \text{ where } A_i \in \mathcal{M} \text{ for all } i.$$

Then for $s$ we define integration in the natural way: If $E \in \mathcal{M}$ then

$$\int_E s \, d\mu \overset{\text{Def.}}{=} \sum_{i=1}^{n} a_i \cdot \mu(A_i \cap E). \quad \textbf{(Int. of simple functions)}$$

If $f : X \to [0, \infty]$ is measurable and $E \in \mathcal{M}$ then we define

$$\int_E f \, d\mu \overset{\text{Def.}}{=} \sup \left\{ \int_E s \, d\mu \mid s \text{ simple }, 0 \le s \le f \right\}. \quad \textbf{(Int. of pos. measurable functions)}$$

This integral is called the **Lebesgue integral of $f$ over $E$** with respect to the measure $\mu$. Its value is in $[0, \infty]$.

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The following propositions are immediate consequences of the definitions.

**Proposition 4** Let $f, g : (X, \mathcal{M}) \to [0, \infty]$ be measurable functions and $E \in \mathcal{M}$. Then

a) If $f \leq g$, then $\int_E f\, d\mu \leq \int_E g\, d\mu$ **(Monotonicity for functions)**

b) If $A \subset B$, then $\int_A f\, d\mu \leq \int_B f\, d\mu$. **(Monotonicity for sets)**

c) If $0 \leq c < \infty$ is a constant, then $\int_E c \cdot f\, d\mu = c \cdot \int_E f\, d\mu$.

d) If $f|_E = 0$, then $\int_E f\, d\mu = 0$ even if $\mu(E) = \infty$.
If $\mu(E) = 0$ then $\int_E f\, d\mu = 0$ even if $f|_E = \infty$ $\;(\ 0 \cdot \infty = \infty \cdot 0 = 0\ )$.

e) $\int_E f\, d\mu = \int_X f \cdot \mathbb{1}_E\, d\mu$

**proof** We only prove a) and b) and leave the rest as an exercise.

a) We recall that

$$\int_E f\, d\mu = \sup\left\{\int_E s\, d\mu \mid s \text{ simple }, 0 \leq s \leq f\right\}.$$

By definition of $g$ we know that $s \leq f \leq g \Rightarrow s \leq g$ hence

b) If $A \subset B$, we note that for any measurable simple function $s : X \to [0, \infty)$ we have that

$$\int_A s\, d\mu = \sum_{i=1}^{n} a_i \cdot \mu(A_i \cap A) =$$

To later prove the additivity of the integral for functions, we first prove it for nonnegative simple functions.

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**Proposition 5** Let $s, t : (X, \mathcal{M}) \to [0, \infty)$ be two nonnegative simple functions (nnsfs). For $E \in \mathcal{M}$ we define

$$\varphi(E) = \int_E s \, d\mu.$$

Then $\varphi$ is a measure on $\mathcal{M}$ and

$$\int_X s + t \, d\mu = \int_X s \, d\mu + \int_X t \, d\mu. \qquad \textbf{(Additivity of integration for nnsfs)}$$

**proof** We know that $s = \sum_{i=1}^n a_i \cdot \mathbb{1}_{A_i}$, where $0 < a_i < \infty$. Furthermore $\varphi(\emptyset) = 0 < \infty$. It remains to show that $\varphi$ is countably additive. Let $(B_k)_{k \in \mathbb{N}}$ be a collection of mutually disjoint elements of $\mathcal{M}$ and $B = \biguplus_{k \in \mathbb{N}} B_k$, then

$$\varphi(B) =$$

To prove the second part let $t = \sum_{j=1}^m c_j \cdot \mathbb{1}_{C_j}$, where $0 < c_j < \infty$. We consider $t$ and $s$ on a common refinement: set $E_{ij} = A_i \cap C_j$. Then

**Picture**

As the statement is true on all $(E_{ij})_{i,j}$, it is true on $X$ by the first part of the proposition. $\square$

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The definition of measurable sets and measures allows us to easily deal with limits.

**Theorem 6 (Lebesgue's Monotone Convergence Theorem (MCT))** Let $(f_n)_{n\in\mathbb{N}} : (X, \mathcal{M}) \to [0, \infty]$ be a sequence of measurable functions on $X$ such that for all $x \in X$

a) $0 \leq f_1(x) \leq f_2(x) \leq \ldots \leq \infty$

b) $\lim_{n\to\infty} f_n(x) = f(x)$ i.e. $f_n \overset{n\to\infty}{\to} f$ pointwise.

Then $f$ is measurable and $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$.

**proof** We note that by **Ch.1.3 Theorem 4** $f = \sup_n f_n$ is measurable and therefore integrable. We show:

**1.)** $(\int_X f_n \, d\mu)_n$ has a limit which is smaller than $\int_X f \, d\mu$

Let $I_n = \int_X f_n \, d\mu$. Since by **Proposition 4 a)** we know that

we know that $(I_n)_n$ is an increasing sequence which attains its limit $I$ in $[0, \infty]$, i.e.

$$\boxed{\lim_{n\to\infty} \int_X f_n \, d\mu = \lim_{n\to\infty} I_n = I} \quad \text{and} \tag{1}$$

As $f_n \leq f$ for all $n \in \mathbb{N}$ we know, again by **Proposition 4 a)** that

It remains to show:

**2.)** $\int_X f \, d\mu \leq I = \lim_{n\to\infty} \int_X f_n \, d\mu$

**Idea:** $\int_X f \, d\mu = \sup\{\int_X s \, d\mu \mid s \text{ simple}, 0 \leq s \leq f\}$. We have to look at those simple functions.
Let $s : X \to [0, \infty)$ be a simple measurable functions, such that $0 \leq s \leq f$ and let $c \in (0, 1)$ be a fixed constant. We define for all $n \in \mathbb{N}$

$$E_n = \{x \in X \mid f_n(x) \geq c \cdot s(x)\}. \quad \text{Then}$$

- $E_n$ is measurable.
- $E_1 \subset E_2 \subset \ldots$ as $(f_n)_n$ is an increasing sequence of functions.
- $X = \bigcup_{n\in\mathbb{N}} E_n$ as for a fixed $x \in X$ we have that $f(x) > c \cdot s(x)$ and $\lim_{n\to\infty} f_n(x) = f(x)$.

Furthermore for $\varphi(E) = \int_E s\, d\mu$ we have as in **Proposition 5**:

As $\varphi$ is a measure we have by **Ch.1.5 Theorem 2 d)** for the left hand side and the fact $X = \bigcup_{n\in\mathbb{N}} E_n$: for all $c \in (0,1)$:

As this is true for all $0 \leq s \leq f$ it is true for the supremum

Hence in total we have proven our claim. □