Math 103: Measure Theory and Complex Analysis
Fall 2018

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Lecture 7

We now prove a convergence theorem for complex valued functions:

Theorem 7 (Lebesgue's Dominated Convergence Theorem (DCT))
Let $(f_n)_{n\in\mathbb{N}} : (X, \mathcal{M}) \to \mathbb{C}$ be a sequence of measurable functions on X such that for all $x \in X$

a) $\lim_{n\to\infty} f_n(x) = f(x)$, i.e. the sequence converges pointwise

b) $|f_n(x)| \le g(x)$ for all $n \in \mathbb{N}$, where $g : (X, \mathcal{M}) \to \mathbb{R}_0^+$ and $g \in \mathcal{L}^1(\mu)$.

Then $f \in \mathcal{L}^1(\mu)$ and

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Picture:

proof Idea: To use the MCT or in this case Fatou's lemma we have to change this into a problem about positive functions.
We know: f is measurable and $|f| \le g$, so $f \in \mathcal{L}^1(\mu)$. Additionally by the $\triangle \neq$ we know that $|f_n - f| \le 2g$. Consider the sequence $(g_n)_{n\in\mathbb{N}}$ where

$$\boxed{g_n = 2g - |f_n - f| \ge 0.} \quad \text{Then} \quad \boxed{\liminf_{n\in\mathbb{N}} g_n = \lim_{n\to\infty} g_n = 2g.}$$

We can now apply Fatou's lemma:

$$\int_X 2g \, d\mu =$$

In total we have that $0 \geq \limsup_{n\in\mathbb{N}} \int_X |f_n - f| \, d\mu$ and we conclude

For the second inequality we use **Theorem 6**. We know

$$\int_X |f_n - f|$$

Hence $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$ □

Example We can now easily prove the statement in **Example 9** of **Lecture 1**: Let $(f_n))_{n\in\mathbb{N}} : [0,1] \to [0,1]$ be a sequence of continuous and therefore measurable functions and suppose that $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$. Then

$$\lim_{n\to\infty} \int_{[0,1]} f_n \, d\mu = \int_{[0,1]} 0 \, d\mu = 0.$$

This follows immediately from the DCT with $g(x) = 1$ for all $x \in [0,1]$.
Picture

Ch. 1.8. Sets of measure zero

Outline If two functions f and g differ in their values only on a set of measure zero, then they are indistinguishable in terms of integration. We say that $f = g$ almost everywhere and can define the class $[f]$ of f this way.

We start with a proposition which we will later apply to sets of measure zero.

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Proposition 1 (Subadditivity) Let $(X, \mathcal{M}, \mu)$ be a measure space and $(A_i)_{i\in\mathbb{N}} \subset \mathcal{M}$. Then

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) \leq \sum_{i\in\mathbb{N}} \mu(A_i).$$

We say that μ is **countably subadditive**.

proof Idea: We subdivide $A = \bigcup_{i\in\mathbb{N}} A_i$ into mutually disjoint measurable subsets.
Picture

Hence as μ is countably additive we have

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \mu\left(\biguplus_{i\in\mathbb{N}} B_i\right) = \sum_{i\in\mathbb{N}} \mu(B_i) \leq \sum_{i\in\mathbb{N}} \mu(A_i). \quad \square$$

Corollary 2 If $(X, \mathcal{M}, \mu)$ is a measure space and for all $i \in \mathbb{N}$: $A_i \subset \mathcal{M}$ and $\mu(A_i) = 0$. Then $\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = 0$.

If $f, g : (X, \mathcal{M}) \to \mathbb{C}$ are measurable. Then

$$S := \{x \in X \mid f(x) \neq g(x)\} = X \backslash (f-g)^{-1}(0) \in \mathcal{M}\,.$$

Definition 3 ($f = g$ almost everywhere) We say that $f = g$ **almost everywhere** if $\mu(S) = 0$. We often write shortly $f = g$ **a. e.** ($[\mu]$).

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Lemma 4 If $f, g : (X, \mathcal{M}) \to \mathbb{C}$ are measurable. Then $f \sim g \Leftrightarrow (f = g$ almost everywhere) is an equivalence relation. Furthermore for any $A \in \mathcal{M}$

$$f \sim g \Rightarrow \int_A |f - g| \, d\mu = 0.$$

proof 1.) Equivalence relation: Clearly by the symmetry of the definition $f \sim f$ and $f \sim g \Leftrightarrow g \sim f$. It remains to show that $f \sim g$ and $g \sim h$ implies that $f \sim h$. Consider the sets

$$\begin{aligned} S_{fg} &= \{x \in X \mid f(x) \neq g(x)\}, \\ S_{gh} &= \{x \in X \mid g(x) \neq h(x)\} \text{ and} \\ S_{fh} &= \{x \in X \mid f(x) \neq h(x)\} \end{aligned}$$

As $f(x) = g(x)$ and $g(x) = h(x)$ implies $f(x) = h(x)$, we have that

2.) Integral: Furthermore if $f \sim g$ then we have that $A = S_{fg} \uplus A \backslash S_{fg}$. Hence

$$\int_A |f - g| \, d\mu =$$

Problem A measurable subset $A \subset B$ of a set B of measure zero has measure zero. However the subset might not be measurable.

Question Can we fix this?
Answer Yes, we can complete the measure to include the subsets of measures zero.

Definiton 5 A measure space $(X, \mathcal{M}, \mu)$ is said **complete** if every subset A of a measurable set B of measure zero is measurable.

Picture

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Theorem 6 Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\mathcal{M}^*$ be the collection of all subsets $A \subset X$, such that there are $F, G \in \mathcal{M}$, such that

$$F \subset A \subset G \quad \text{and} \quad \mu(G \backslash F) = 0.$$

Then $\mathcal{M}^*$ is a σ algebra, such that $\mathcal{M} \subset \mathcal{M}^*$.
We set

$$\mu^*(A) \overset{\text{Def.}}{:=} \mu(F) \ \text{ for all } \ A \in \mathcal{M}^*\,.$$

Then $(X, \mathcal{M}^*, \mu^*)$ is a measure space and $(X, \mathcal{M}^*, \mu^*)$ is complete.

proof We prove the statement in three steps.

1.) μ^* is well-defined on $\mathcal{M}^*$

Suppose that

$$F \subset A \subset G \ \text{ and } \ F' \subset A \subset G', \ \text{ such that } \ \mu(G \backslash F) = \mu(G' \backslash F') = 0$$

We have to show that $\mu(F) = \mu(F') = \mu(A)$. We know that

□

2.) $\mathcal{M}^*$ is a σ algebra

We check the three conditions for a σ algebra.

a) $\emptyset, X \in \mathcal{M}^*$ as $\emptyset, X \in \mathcal{M}$.

b) $A \subset \mathcal{M}^* \Rightarrow A^c \in \mathcal{M}^*$:

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c) $(A_i)_{i\in\mathbb{N}} \subset \mathcal{M}^* \Rightarrow \bigcup_{i\in\mathbb{N}} A_i \in \mathcal{M}^*$: As $(A_i)_{i\in\mathbb{N}} \subset \mathcal{M}^*$ we know that for all $i \in \mathbb{N}$

In total a) - c) imply that $\mathcal{M}^*$ is a σ algebra. □

3.) μ^* is a measure

As $\mu(\emptyset) = \mu^*(\emptyset) = 0$ it remains to show that μ^* is countably additive. $(A_i)_{i\in\mathbb{N}} \subset \mathcal{M}^*$ be sets in $\mathcal{M}^*$ that are mutualy disjoint. For $A = \biguplus_{i\in\mathbb{N}} A_i \in \mathcal{M}^*$ we have to show that

$$\mu^*(\biguplus_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} \mu^*(A_i).$$

As $(A_i)_{i\in\mathbb{N}} \subset \mathcal{M}^*$ we know that for all $i \in \mathbb{N}$

Hence μ^* is a measure. □