Math 103: Measure Theory and Complex Analysis
Fall 2018

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Lecture 8

We now show that if the measure space is complete, all functions in the equivalence class of a measurable function are measurable.

Lemma 7 Let $(X, \mathcal{M}, \mu)$ be a complete measure space and $f, g : (X, \mathcal{M}) \to \mathbb{C}$ be functions, such that g is measurable and $f \sim g$. Then f is also measurable.

proof Let $S = \{x \in X \mid f(x) \neq g(x)\}$ then $S^c = \{x \in X \mid f(x) = g(x)\}$.
If $V \subset \mathbb{C}$ is an open set. Then

$$f^{-1}(V) =$$

As the measure is complete $f^{-1}(V) \cap S$ is measurable as the subset of a set of measure zero. This means that $f^{-1}(V)$ is measurable for every open set $V \subset \mathbb{C}$. Hence f is measurable. $\square$

Corollary 8 Suppose $(X, \mathcal{M}, \mu)$ is a complete measure space and $(f_n)_{n \in \mathbb{N}} : (X, \mathcal{M}) \to \mathbb{C}$ be a sequence of measurable functions on X. If $f : (X, \mathcal{M}) \to \mathbb{C}$ is a function, such that

$$\lim_{n \to \infty} f_n(x) = f(x) \text{ for almost all } x \in X$$

then f is measurable.

proof Set $g = \limsup_{n \in \mathbb{N}} \mathrm{Re}(f_n) + i \cdot \limsup_{n \in \mathbb{N}} \mathrm{Im}(f_n)$.

$\square$

Proposition 9 Let $(X, \mathcal{M}, \mu)$ be a complete measure space.

a) If $f : (X, \mathcal{M}) \to [0, \infty]$ is measurable and $\int_A f \, d\mu = 0$, where $A \in \mathcal{M}$ then $f = 0$ for almost all $x \in A$.

b) If $f \in \mathcal{L}^1(\mu)$ and $\int_A f \, d\mu = 0$ for all $A \in \mathcal{M}$ then $f = 0$ almost everywhere.

proof a) Let $A_n = \{x \in A \mid f(x) \geq \frac{1}{n}\}$. Then

$$\frac{1}{n}\mu(A_n) \leq$$

It follows that $\{x \in A \mid f(x) \neq 0\} = \{x \in A \mid f(x) > 0\} = \bigcup_{n \in \mathbb{N}} A_n$. By the subbadditivity of the measure we have that $\mu(\bigcup_{n \in \mathbb{N}} A_n) = 0$. Hence $f = 0$ for almost all $x \in A$. $\square$

b) Let $f = u + iv$ and set $U^+ = \{x \in X \mid u(x) \geq 0\} \in \mathcal{M}$. Then

$$\int_{U^+} f \, d\mu =$$

In a similar fashion we show that u^-, v^+ and v^- are zero almost everywhere. $\square$

Ch. 1.9. Outer measure

Outline In the previous part we started with a σ algebra $\mathcal{M} \subset \mathcal{P}(X)$ and defined a measure μ on it. In this part we will construct an **outer measure** that is defined on all subsets of X but is not a true measure, and then construct a actual measure by restricting the outer measure to an appropriate σ algebra of measurable sets.

We will start with the definition of the outer measure.

Definition 1 (Outer measure) An **outer measure** on a set X is a function $\mu^o : \mathcal{P}(X) \to [0, \infty]$, such that

a) $\mu^o(\emptyset) = 0$.

b) $A \subset B \Rightarrow \mu^o(A) \leq \mu^o(B)$. **(Monotonicity)**

c) $\mu^o \left(\bigcup_{i \in \mathbb{N}} A_i \right) \leq \sum_{i \in \mathbb{N}} \mu^o(A_i)$. **(Countable subadditivity)**

We now define "measurable sets" and show that they form a σ algebra for μ^o meaning that they are indeed measurable in the original sense.

Definition 2 (measurable sets) Let $\mu^o : \mathcal{P}(X) \to [0, \infty]$ be an outer measure. We say $A \subset X$ is μ^o **measurable** if

$$\boxed{\mu^o(B) = \mu^o(A \cap B) + \mu^o(A^c \cap B) \quad \text{for all} \quad B \subset X}.$$

We set $\mathcal{M}^o = \{A \subset X \mid A \text{ is } \mu^o \text{ measurable}\}$.
Picture

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Remark 3 Since μ^o is finitely subadditive and $B = (A \cap B) \cup (A^c \cap B)$ we know that $\mu^o(B) \leq \mu^o(A \cap B) + \mu^o(A^c \cap B)$. Hence we have

$$\boxed{A \in \mathcal{M}^o \Leftrightarrow \mu^o(B) \geq \mu^o(A \cap B) + \mu^o(A^c \cap B) \text{ for all } B \subset X.}$$

This inequality is fulfilled if $\mu^o(B) = \infty$. Hence we can restrict ourselves to the sets $B \subset X, \mu^o(B) < \infty$ in the above inequality.

Theorem 5 Let $\mu^o : \mathcal{P}(X) \to [0, \infty]$ be an outer measure. Then $\mathcal{M}^o$ is a σ algebra on X and $\mu \overset{\text{Def.}}{=} \mu^o|_{\mathcal{M}^o}$ is a complete measure on $(X, \mathcal{M}^o)$.

proof We proceed in three steps.

1.) $\mathcal{M}^o$ is almost a σ algebra

We show the first two properties of a σ algebra and that $\mathcal{M}^o$ is closed under finite unions.

a) Clearly for $A = \emptyset$ in the definition of a measurable set

$$\mu^o(B) =$$

Hence $\emptyset \in \mathcal{M}^o$ and by the symmmetry of the definition $X \in \mathcal{M}^o$.

b) We have to show that $A \in \mathcal{M}^o \Rightarrow A^c \in \mathcal{M}^o$: This follows again by the symmetry of the definition of $\mathcal{M}^o$ with respect to complements.

c) Closure under finite union: We have to show: $A_1, A_2 \in cM^o \Rightarrow A_1 \cup A_2 \in \mathcal{M}^o$. Fix $B \subset X$. We know, as $A_1, A_2 \in \mathcal{M}^o$:

$$\begin{aligned} \mu^o(B) &= && \text{and} \\ \mu^o(B \cap A_1^c) &\overset{A_2 \in \mathcal{M}^o}{=} && . \\ \mu^o(B) &= \mu^o(A_1 \cap B) + \mu^o(B \cap A_1^c \cap A_2) + \mu^o(B \cap A_1^c \cap A_2^c) \\ &\overset{\mu^o \text{ subadd.}}{\geq} \\ &= && . \end{aligned}$$

The last equation is true as $B \cap (A_1 \cup A_2) = (B \cap A_1) \cup (B \cap A_2 \backslash A_1) = (B \cap A_1) \cup (B \cap A_2 \cap A_1^c)$. The finite subadditivity then follows by induction.